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The transmission factor $T_1(N,N)$ revisited

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Abstract

The effect of two non-extended dead times in series on the count rate of an incoming Poisson process is studied anew. Compared to an earlier approach to the same problem, much simpler results are obtained which can be presented in closed form. Some series expansions are derived for various experimental situations and a proof is given ensuring the exact equivalence between the old and the new expressions.

1. Introduction

For a series arrangement of two dead times, it has become usual to describe the overall effect on the count rates by the relation

$$R = \rho \; T_2 \cdot T_1 , \tag{1}$$

where T_2 and T_1 are so-called transmission factors. They can be defined as follows (cf. Fig. 1): if R_o is the output rate R for $\tau_1 = 0$, i.e. in the absence of a first dead-time element, then

$$T_2 = \frac{R_o}{\rho} \quad \text{and} \quad T_1 = \frac{R}{R_o} . \tag{2}$$

Hence, for a given input process, T_2 depends only on τ_2 while T_1 accounts for the additional influence of τ_1. For the sake of simplicity (and in agreement with the usual experimental situation) we assume that the input process, of rate ρ, is Poissonian.

For the two traditional types of dead time, namely non-extended (N) and extended (E), four different combinations are possible for an arrangement of two elements in series. Obviously, they all lead to different expressions for T_1.

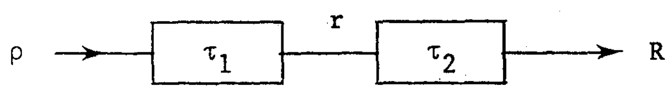

Fig. 1 - Schematic arrangement of two dead times in series (with $\tau_1 < \tau_2$) and the corresponding count rates.

It turns out [1] that, for a Poisson input, two of them, namely those corresponding to the type sequences "N,E" and "E,N", lead to very simple formulae whereas the arrangements in which the two dead times are of the same type are more difficult to describe. By far the most complicated expression has been obtained for the sequence "N,N"; in hindsight, therefore, we find it somewhat astonishing that it is exactly this case that has been first analyzed in a rigorous way [2].

We suggest to come back once more to this problem for two reasons. The first is that we can now propose a new and simpler formula. The result, although equivalent to the previous one, is much easier to use. The second reason is due to the fact that certain series developments have become of interest recently as they can be compared directly with possible generalizations where one or both dead times are of the Albert-Nelson type. The new formula allows us to derive the necessary power series in a much simpler and more reliable way.

2. The new evaluation of $T_1(N,N)$

In contrast to the approach suggested by (1), we now determine the counting losses in the same order as they occur, i.e. first the effect of τ_1 and then the additional losses due to τ_2. The transmission T_1 as defined above will be obtained in a later step.

For a Poisson input, the transmission factor for τ_1 alone, which we denote here by $\tilde{T}_1$, is given by

$$\tilde{T}_1 = \frac{1}{1+\rho\tau_1}, \tag{3}$$

so that the original count rate ρ is reduced to $r = \rho\,\tilde{T}_1$ (cf. Fig. 1).

After the first non-extended dead time the interval density is known to be given by

$$f(t) = \rho\, e^{-\rho(t-\tau_1)}, \qquad \text{for} \quad t > \tau_1 ,$$

which yields for the k-fold density (k = 1, 2, ...)

$$\begin{aligned} f_k(t) &= \{f(t)\}^{*k} \\ &= \frac{[\rho(t-k\tau_1)]^{k-1}}{(k-1)!}\, e^{-\rho(t-k\tau_1)}, \qquad \text{for} \quad t > k\tau_1 . \end{aligned}$$

The cumulative or total arrival density of pulses after τ_1, with an event at t = 0, is given by the sum

$$D(t) = \sum_{k=1}^{K} f_k(t) ,$$

where K is defined by $K\tau_1 < \tau_2 \leq (K+1)\,\tau_1$ or, in the notation used previously, by $K = [[\tau_2/\tau_1]]$.

We now have to evaluate the (additional) losses produced by the second dead time τ_2. Since every registered output event (R) is followed by τ_2, each one has to be associated with a loss probability

$$\Pi_K = \int_0^{\tau_2} D(t)\,dt$$

$$= \sum_{k=1}^{K} \rho \int_{k\tau_1}^{\tau_2} \frac{[\rho(t-k\tau_1)]^{k-1}}{(k-1)!}\, e^{-\rho(t-k\tau_1)}\,dt$$

$$= \rho \sum_{k=1}^{K} \frac{1}{(k-1)!} \int_0^{\tau_2-k\tau_1} (\rho t)^{k-1}\, e^{-\rho t}\,dt\ . \tag{4}$$

According to Dwight [3] we have, for m = 0, 1, 2, ...,

$$\int_0^x t^m\, e^{-\alpha t}\,dt = \frac{m!}{\alpha^{m+1}} - e^{-\alpha x} \sum_{j=0}^{m} \frac{m!}{(m-j)!}\, \frac{x^{m-j}}{\alpha^{j+1}}\ , \tag{5}$$

which in our case leads to

$$\Pi_K = \sum_{k=1}^{K} \frac{\rho^k}{(k-1)!} \left\{ \frac{(k-1)!}{\rho^k} - e^{-\rho(\tau_2-k\tau_1)} \sum_{j=0}^{k-1} \frac{(k-1)!}{(k-1-j)!}\, \frac{(\tau_2-k\tau_1)^{k-1-j}}{\rho^{j+1}} \right\}$$

or, after simplifying, to

$$\Pi_K = \sum_{k=1}^{K} \left\{ 1 - e^{-(1-k\alpha)x} \sum_{j=0}^{k-1} \frac{[(1-k\alpha)x]^j}{j!} \right\}\ , \tag{6}$$

where we have put, as usual, $\rho\tau_2 = x$ and $\tau_1/\tau_2 = \alpha$.

From the count-rate balance

$$R = r - R\,\Pi_K$$

we find, putting $R = r\,{}_K\tilde{T}_2$, for the second new transmission factor

$${}_K\tilde{T}_2 = \frac{1}{1 + \Pi_K}\ , \tag{7}$$

where Π_K is given by (6).

How can we arrive from this at the requested tansmission factor T_1? Since the output rate may also be written in the form

$$R = \rho \; \tilde{T}_1 \cdot {}_K\tilde{T}_2 \; , \tag{8}$$

a simple comparison with (1) yields for $T_1 \equiv {}_KT_1(N,N)$ the relation

$${}_KT_1(N,N) = \frac{\tilde{T}_1 \cdot {}_K\tilde{T}_2}{T_2(N)} \; , \tag{9}$$

with $\tilde{T}_1$ given by (3) and ${}_K\tilde{T}_2$ by (7). Since obviously

$$T_2(N) = \frac{1}{1 + \rho\tau_2} \; ,$$

we readily find for the transmission factor the formula

$$\begin{aligned} {}_KT_1(N,N) &= \frac{1 + \rho\tau_2}{(1 + \rho\tau_1)\,(1 + \Pi_K)} \\ &= \frac{1 + x}{1 + \alpha x} \left\{1 + \sum_{k=1}^{K} \left[1 - e^{-(1-k\alpha)x} \sum_{j=0}^{k-1} \frac{[x(1-k\alpha)]^j}{j!}\right]\right\}^{-1} . \end{aligned} \tag{10}$$

This is the main result of the present study.

It is not obvious, though nevertheless true, that expression (10) is identical with the more complicated one given previously. This point will be elaborated in Appendix C and proved in Appendix D.

4. Some series developments

For many applications it is useful to have approximations of ${}_KT_1(N,N)$ available, usually in the form of series expansions. As their actual derivation is rather cumbersome, we shall limit ourselves to illustrating the procedure and giving a selection of the intermediate results. Apart from care and patience nothing particular is required for obtaining them.

As, according to (10), T_1 depends on the range of α, we have to consider the different values of K individually. There is, however, in the general expression

$${}_KT_1(N,N) = \frac{1 + x}{1 + \alpha x} \cdot {}_K\tilde{T}_2 \tag{11}$$

a common factor which we develop first, namely

$$C \equiv \frac{1+x}{1+\alpha x} = 1 + (1-\alpha)\, x \sum_{j=0}^{\infty} (-\alpha x)^j$$
$$\cong 1 + (1-\alpha)\, x - \alpha(1-\alpha)\, x^2 + \alpha^2(1-\alpha)\, x^3 - \alpha^3(1-\alpha)\, x^4 \; . \tag{12}$$

a) For $\underline{K = 1,}$ i.e. $1/2 \leqslant \alpha \leqslant 1$,

it follows from (6) that

$$\Pi_1 = 1 - e^{-(1-\alpha)x} \; . \tag{13}$$

A development up to fourth order gives

$$\Pi_1 \cong (1-\alpha)\, x - \frac{1}{2}(1-\alpha)^2\, x^2 + \frac{1}{6}(1-\alpha)^3\, x^3 - \frac{1}{24}(1-\alpha)^4\, x^4 \tag{13'}$$

and its inversion, after a number of rearrangements, yields

$$\begin{aligned} {}_1\tilde{T}_2 &= (1+\Pi_1)^{-1} \\ &\cong 1 - (1-\alpha)\, x + \frac{1}{2}(2-2\alpha-\alpha^2)\, x^2 - \frac{1}{6}(5+3\alpha-33\alpha^2+27\alpha^3)\, x^3 \\ &\quad + \frac{1}{24}(14+66\alpha-414\alpha^2+572\alpha^3-251\alpha^4)\, x^4 \; . \end{aligned}$$

Thus, with (12) we finally obtain from (11) for the transmission factor (in the case K = 1), up to fourth order, the expansion

$$\begin{aligned} {}_1T_1(N,N) &\cong 1 + \frac{1}{2}(1-\alpha)(1-3\alpha)\, x^2 - \frac{1}{3}(1-\alpha)(2-7\alpha+2\alpha^2)\, x^3 \\ &\quad + \frac{1}{24}(1-\alpha)(23-105\alpha+117\alpha^2-59\alpha^3)\, x^4 \; . \end{aligned} \tag{14}$$

b) For $\underline{K = 2,}$ i.e. for $1/3 \leqslant \alpha \leqslant 1/2$,

equation (6) leads readily to

$$\Pi_2 = 2 - e^{-(1-\alpha)x} - e^{-(1-2\alpha)x}\left[1 + (1-2\alpha)x\right] \; . \tag{15}$$

As before, this has to be expanded to

$$\begin{aligned} \Pi_2 &\cong (1-\alpha)\, x - \frac{1}{2}\alpha(2-3\alpha)\, x^2 - \frac{1}{6}(1-9\alpha+21\alpha^2-15\alpha^3)\, x^3 \\ &\quad + \frac{1}{24}(2-20\alpha+66\alpha^2-92\alpha^3+47\alpha^4)\, x^4 \; , \end{aligned} \tag{15'}$$

and by an inversion we can arrive for $(1 + \Pi_2)^{-1}$ at the expression

$$
{}_2\tilde{T}_2 \cong 1 - (1-\alpha)\, x + \frac{1}{2}\,(2-2\alpha-\alpha^2)\, x^2 - \frac{1}{6}\,(5+3\alpha-33\alpha^2+27\alpha^3)\, x^3 + \frac{1}{24}\,(14+76\alpha-390\alpha^2+500\alpha^3-197\alpha^4)\, x^4 \, .
$$

Multiplication by C finally leads to

$$
{}_2T_1(N,N) \cong 1 - \frac{1}{2}\,\alpha^2\, x^2 + \frac{1}{6}\,(1-9\alpha+30\alpha^2-24\alpha^3)\, x^3 - \frac{1}{24}\,(6-60\alpha+222\alpha^2-272\alpha^3+101\alpha^4)\, x^4 \, . \tag{16}
$$

For larger values of the integer K the calculations to be performed for arriving at the transmission factor T_1 are very similar to those sketched above, although increasingly longer. We give the explicit results up to fourth order in Table 1.

It is interesting to compare the various series expansions of ${}_KT_1(N,N)$ for increasing values of K. We observe that all the contributions of order x^k are the same, provided that $k < K$: they lie below the dashed line in Table 1. However, the terms of order k beyond K show no simple pattern. This behaviour can be fairly well understood by a closer look at the quantities Π_K, which we shall take in Appendix A, and their effect on T_1 (Appendix B).

5. Evaluation of the forms ${}_KT_1^{-1}(N,N)$

In various applications the quantity needed is the reciprocal of T_1 rather than T_1 itself. There is, indeed, no good reason to prefer one or the other form. In addition, the series expansions might turn out to be somewhat simpler for T_1^{-1}.

The numerical determination of the reciprocal series (always to fourth order) is straightforward and the results are assembled in Table 2.

For lengthy calculations simple checks are always welcome, even if the control is not a complete one. In the case of our series developments, this can be easily done by choosing for α the limiting value where two adjacent regions (specified by K) join. Thus, for $\alpha = 1/2$ we find indeed that

Table 1 - Series expansions of the transmission factors

$${}_K T_1(N,N) \;\cong\; 1 + a_2 x^2 + a_3 x^3 + a_4 x^4$$

K	a_2	a_3	a_4
1	$\frac{1}{2}(1-\alpha)(1-3\alpha)$	$-\frac{1}{3}(1-\alpha)(2-7\alpha+2\alpha^2)$	$\frac{1}{24}(1-\alpha)(23-105\alpha+117\alpha^2-59\alpha^3)$
2	$-\frac{1}{2}\alpha^2$	$\frac{1}{6}(1-9\alpha+30\alpha^2-24\alpha^3)$	$-\frac{1}{24}(6-60\alpha+222\alpha^2-272\alpha^3+101\alpha^4)$
3	$-\frac{1}{2}\alpha^2$	$\frac{1}{2}\alpha^2(1+\alpha)$	$\frac{1}{24}(1-16\alpha+84\alpha^2-268\alpha^3+250\alpha^4)$
$\geqslant 4$	$-\frac{1}{2}\alpha^2$	$\frac{1}{2}\alpha^2(1+\alpha)$	$-\frac{1}{4}\alpha^2(2+2\alpha+\alpha^2)$

Table 2 - Series expansions of the reciprocal transmission factors

$${}_K T_1^{-1}(N,N) \;\cong\; 1 + b_2 x^2 + b_3 x^3 + b_4 x^4$$

K	b_2	b_3	b_4
1	$-\frac{1}{2}(1-\alpha)(1-3\alpha)$	$\frac{1}{3}(1-\alpha)(2-7\alpha+2\alpha^2)$	$-\frac{1}{24}(1-\alpha)(17-63\alpha+27\alpha^2-5\alpha^3)$
2	$\frac{1}{2}\alpha^2$	$-\frac{1}{6}(1-9\alpha+30\alpha^2-24\alpha^3)$	$\frac{1}{24}(6-60\alpha+222\alpha^2-272\alpha^3+107\alpha^4$
3	$\frac{1}{2}\alpha^2$	$-\frac{1}{2}\alpha^2(1+\alpha)$	$-\frac{1}{24}(1-16\alpha+84\alpha^2-268\alpha^3+244\alpha^4)$
$\geqslant 4$	$\frac{1}{2}\alpha^2$	$-\frac{1}{2}\alpha^2(1+\alpha)$	$\frac{1}{2}\alpha^2(1+\alpha+\alpha^2)$

$$ {}_1T_1 = {}_2T_1 \cong 1 - \frac{1}{8}x^2 + \frac{1}{6}x^3 - \frac{61}{384}x^4 $$

and (17a)

$$ {}_1T_1^{-1} = {}_2T_1^{-1} \cong 1 + \frac{1}{8}x^2 - \frac{1}{6}x^3 + \frac{67}{384}x^4 \; . $$

Likewise one obtains for $\alpha = 1/3$

$$ {}_2T_1 = {}_3T_1 \cong 1 - \frac{1}{18}x^2 + \frac{2}{27}x^3 - \frac{149}{1\,944}x^4 $$

(17b)

$$ {}_2T_1^{-1} = {}_3T_1^{-1} \cong 1 + \frac{1}{18}x^2 - \frac{2}{27}x^3 + \frac{155}{1\,944}x^4 \; , $$

and for $\alpha = 1/4$

$$ {}_3T_1 = {}_4T_1 \cong 1 - \frac{1}{32}x^2 + \frac{5}{128}x^3 - \frac{41}{1\,024}x^4 $$

(17c)

$$ {}_3T_1^{-1} = {}_4T_1^{-1} \cong 1 + \frac{1}{32}x^2 - \frac{5}{128}x^3 + \frac{21}{512}x^4 \; . $$

Hence all the expressions listed in Tables 1 and 2 pass this test successfully.

6. The limiting transmission factor

The formulae listed in Tables 1 and 2 do not go beyond fourth order in x, but one might wonder how these developments continue for larger values of K, i.e. for $\tau_1 \ll \tau_2$.

A look at the coefficients in Table 1 does not allow us to make a reliable prediction for the higher coefficients a_k. On the other hand, there might be some hope for the coefficients b_k appearing in Table 2 for the reciprocal series. This conjecture seems worth checking.

Let us decide on order 6. After rather tedious developments we arrive for Π_6 at the expansion

$$ \Pi_6 \cong (1-\alpha)\,x - \frac{1}{2}(2\alpha-3\alpha^2)\,x^2 + (\alpha^2-2\alpha^3)\,x^3 - \frac{1}{2}(2\alpha^3-5\alpha^4)\,x^4 + (\alpha^4-3\alpha^5)\,x^5 - \frac{1}{2}(2\alpha^5-7\alpha^6)\,x^6 \; . \qquad (18) $$

This suggests the general formula, valid up to x^K,

$$ \Pi_K \cong -\frac{1}{2}\sum_{k=1}^{K}\left[2\alpha^{k-1} - (k+1)\alpha^k\right](-x)^k \; , \qquad \text{for} \quad K > 1 \; . \qquad (19) $$

It follows from (10) that

$$ {}_K T_1^{-1}(N,N) = \frac{1}{C}(1+\Pi_K) , \tag{20}$$

with C defined in (12), thus

$$ 1/C = 1 + (1-\alpha) \sum_{j=1}^{\infty} (-x)^j . \tag{21}$$

If (20) is actually evaluated for K = 6, i.e. essentially by multiplying (18) with (21), we arrive at

$$ \begin{aligned} {}_6 T_1^{-1}(N,N) \cong\ & 1 + \frac{1}{2}\alpha^2 x^2 - \frac{1}{2}\alpha^2 (1+\alpha)\, x^3 + \frac{1}{2}\alpha^2 (1+\alpha+\alpha^2)\, x^4 \\ & - \frac{1}{2}\alpha^2 (1+\alpha+\alpha^2+\alpha^3)\, x^5 + \frac{1}{2}\alpha^2 (1+\alpha+\alpha^2+\alpha^3+\alpha^4)\, x^6 . \end{aligned} \tag{22}$$

This strongly suggests, for an arbitrary value of K, the general relation (up to x^K)

$$ {}_K T_1^{-1}(N,N) \cong 1 + \frac{1}{2}\alpha^2 x^2 \sum_{k=0}^{K} (-x)^k \sum_{j=0}^{k} \alpha^j . \tag{23}$$

Of course, the series expansion (22) can now likewise be inverted to yield ${}_6T_1(N,N)$. This is easier than one might fear, for the necessary formulae given explicitly in [4] may be much simplified since the coefficient of x is absent. The result is

$$ \begin{aligned} {}_6 T_1(N,N) \cong\ & 1 - \frac{1}{2}\alpha^2 x^2 + \frac{1}{2}\alpha^2 (1+\alpha)\, x^3 - \frac{1}{2}\alpha^2 \left(1+\alpha+\frac{1}{2}\alpha^2\right) x^4 \\ & + \frac{1}{2}\alpha^2 (1+\alpha)\, x^5 - \frac{1}{2}\alpha^2 \left(1+\alpha-\frac{1}{2}\alpha^2-\alpha^3-\frac{1}{4}\alpha^4\right) x^6 . \end{aligned} \tag{24}$$

If we have confidence in (23), two more terms can be derived by means of [4], namely (for K > 8)

$$ \begin{aligned} & + \frac{1}{2}\alpha^2 \left(1+\alpha-\alpha^2-2\alpha^3-\frac{5}{4}\alpha^4-\frac{1}{4}\alpha^5\right) x^7 \\ & - \frac{1}{2}\alpha^2 \left(1+\alpha-\frac{3}{2}\alpha^2-3\alpha^3-2\alpha^4-\frac{3}{4}\alpha^5-\frac{1}{8}\alpha^6\right) x^8 , \end{aligned} \tag{25}$$

but even now a general law for the formation of the coefficients in the development still does not become visible. This confirms our earlier conjecture that the limiting series expansion is simple only for $T_1^{-1}(N,N)$.

It will be obvious that all the expressions given in this section for $K \gg 1$ are at most of some theoretical, but of virtually no practical, interest. Since $K \to \infty$ implies $\alpha \to 0$, the limiting transmission factor $T_1(N,N)$ will inevitably tend towards unity.

Analogous developments for the three other arrangements of two classical dead times in series will be given in a subsequent report.

APPENDICES

A. Recursion formula for Π_K

It is obvious from (10) that the loss probability, denoted by Π_K, plays a key role in the evaluation of the transmission factor $T_1(N,N)$. How does it change if we decide to increase K by one unit, i.e. for the range of smaller values of α?

This question calls for the elaboration of a recursion formula. It follows from (6) that (for $K \geq 2$)

$$\Pi_K = \Pi_{K-1} + 1 - e^{-(1-K\alpha)x} \sum_{k=0}^{K-1} \frac{[(1-K\alpha)x]^k}{k!} . \tag{A1}$$

By restricting ourselves to terms up to order x^K we can write

$$\begin{aligned}
\Pi_K - \Pi_{K-1} &\cong 1 - \sum_{j=0}^{K} \frac{[-(1-K\alpha)x]^j}{j!} \sum_{k=0}^{K-1} \frac{[(1-K\alpha)x]^k}{k!} \\
&= 1 - \{1 + [-(1-K\alpha)x]^K \sum_{k=0}^{K-1} \frac{(-1)^k}{(K-k)!\ k!} \} \\
&= - [-(1-K\alpha)x]^K \{ \sum_{k=0}^{K} \frac{(-1)^k}{(K-k)!\ k!} - \frac{(-1)^K}{K!} \} \\
&= \frac{(1-K\alpha)^K}{K!} x^K ,
\end{aligned} \tag{A2}$$

as the sum over k in the curly brackets disappears.

This shows that an increase from K to K+1 leaves all the contributions up to x^K unchanged, and this is clearly also true for T_1.

It is even possible to go a step beyond (A2) and to derive a general expression which gives the changes for all the powers of x. We limit ourselves to sketching the reasoning. The starting point is a slight variation of (A1), namely

$$\Pi_K - \Pi_{K-1} = 1 - e^{-(1-K\alpha)x} \left\{ e^{(1-K\alpha)x} - \sum_{k=K}^{\infty} \frac{[(1-K\alpha)x]^k}{k!} \right\}$$

$$= e^{-(1-K\alpha)x} \sum_{k=K}^{\infty} \frac{[(1-K\alpha)x]^k}{k!} \,. \qquad \text{(A3)}$$

Execution of the multiplication leads, after a rearrangement, to

$$\Pi_K - \Pi_{K-1} = \frac{[(1-K\alpha)x]^K}{K!} + \frac{[(1-K\alpha)x]^{K+1}}{(K+1)!} \{1 - (K+1)\}$$

$$+ \frac{[(1-K\alpha)x]^{K+2}}{(K+2)!} \left\{1 - (K+2) + \frac{1}{2}(K+1)(K+2)\right\}$$

$$+ \frac{[(1-K\alpha)x]^{K+3}}{(K+3)!} \left\{1 - (K+3) + \frac{1}{2}(K+2)(K+3) - \frac{1}{6}(K+1)(K+2)(K+3)\right\}$$

$$+ \frac{[(1-K\alpha)x]^{K+4}}{(K+4)!} \left\{1 - (K+4) + \frac{1}{2}(K+3)(K+4) - \frac{1}{6}(K+2)(K+3)(K+4) + \frac{1}{24}(K+1)(K+2)(K+3)(K+4)\right\}$$

$$+ \ldots \,.$$

An evaluation of the curly brackets appearing in the above expression yields, in order,

$$-K \,,$$

$$\frac{1}{2} K (1 + K) \,,$$

$$-\frac{1}{6} K (2 + 3K + K^2) \,,$$

$$\frac{1}{24} K (6 + 11K + 6K^2 + K^3) \,,$$

where we see the Stirling numbers (of first kind) appearing as coefficients. In retrospect, this is not really a surprise. The general recursion formula looked for can thus be written as

$$\Pi_K - \Pi_{K-1} = \sum_{k=0}^{\infty} \frac{[(1-K\alpha)x]^{K+k}}{(K+k)!} \frac{(-1)^k}{k!} \sum_{j=0}^{k} \left| s(k,j) \right| K^j , \qquad \text{(A4)}$$

where we have to put $s(k,0) = \delta_{0,k}$.

When (A4) is applied to K = 2, as an example, we find

$$\Pi_2 - \Pi_1 = \frac{1}{2}(1-2\alpha)^2 x^2 - \frac{1}{3}(1-2\alpha)^3 x^3 + \frac{1}{8}(1-2\alpha)^4 x^4 - \frac{1}{30}(1-2\alpha)^5 x^5 \pm \ldots , \qquad \text{(A4')}$$

which is an agreement with the explicit expressions given in (13') and (15'), as can be easily checked.

B. Updating reciprocal series

It follows from (10) that the transmission factor ${}_K T_1(N,N)$ is proportional to $(1 + \Pi_K)^{-1}$. There thus arises the question of how an increase in K, the influence of which on Π_K has been studied in Appendix A, affects the transmission factor. In other words, we would like to know how a change of some coefficients in a series development modifies the corresponding reciprocal series.

Let us start with an expansion

$$S \equiv 1 + a_1 x + a_2 x^2 + \ldots = 1 + \sum_{j=1}^{\infty} a_j x^j , \qquad \text{(B1)}$$

with known coefficients a_j.

For the reciprocal series

$$1/S \equiv 1 + b_1 x + b_2 x^2 + \ldots = 1 + \sum_{j=1}^{\infty} b_j x^j \qquad \text{(B2)}$$

the new coefficients b_j can be obtained by means of known expressions. These have been explicitly stated in [4] up to order 8; the coefficients b_j are therefore supposed to be known in what follows.

We now assume that S is modified, for instance as the result of improved measurements (or, in our case, a change in K). This can be expressed by

$$\tilde{S} = S + S_c , \qquad \text{with} \quad S_c = \sum_{j=k}^{\infty} c_j x^j , \tag{B3}$$

where S_c describes the change, with known coefficients c_j (up to a certain order).

Hence, in the updated series

$$\tilde{S} = 1 + \sum_{j=1}^{\infty} A_j x^j \tag{B4}$$

we have $\quad A_j = a_j \quad , \quad \text{for} \quad j < k$

$\qquad\qquad A_j = a_j + c_j , \quad \text{"} \quad j > k .$

What is the effect of this modification on the updated reciprocal series? Let us denote it by

$$1/\tilde{S} \equiv 1 + \sum_{j=1}^{\infty} B_j x^j . \tag{B5}$$

We now wish to know how the new coefficients (B_j) are related to the old ones (b_j). Obviously one still has $B_j = b_j$ for $j < k$, but for $j > k$ the connection will be more complicated. For the moment we leave k unspecified; it will be easy to put afterwards $c_j = 0$ for $j < k$.

For the derivation of the general relations which link the new with the old coefficients, we just have to use some formulae derived previously and adapt them to the present context. For the first two coefficients this yields (by means of eq. 4 in [4])

$$B_1 = - A_1 = - (a_1+c_1) = b_1 - c_1 ,$$

$$B_2 = A_1^2 - A_2 = (a_1+c_1)^2 - (a_2+c_2) = b_2 + 2a_1c_1 + c_1^2 - c_2 .$$

For the coefficients of higher order the rearrangements are equally simple, but become increasingly longer. We restrict ourselves to list the final results which may be expressed in the form

$$
\begin{aligned}
B_1-b_1 &= -c_1 ,\\
B_2-b_2 &= c_1(2a_1+c_1) - c_2 ,\\
B_3-b_3 &= -c_1[3a_1(a_1+c_1)-2(a_2+c_2)+c_1^2)] + 2c_2a_1 - c_3 ,\\
B_4-b_4 &= -c_1[6(a_1a_2+a_1c_2-a_2^2c_1)-4a_1(a_1^2+c_1^2)+3c_1(a_1+c_2)-2(a_3+c_3)-c_1^3]\\
&\quad - c_2(3a_1^2-2a_2-c_2) + 2c_3a_1 - c_4 ,\\
B_5-b_5 &= c_1[12a_1(a_1a_2+a_2c_1+a_1c_2+c_1c_2) - 10a_1^2c_1(a_1+c_1)\\
&\quad -6(a_1a_3+a_1c_3+a_2c_2)-5a_1(a_1^3+c_1^3)+4c_1^2(a_2+c_2)\\
&\quad -3(a_3c_1+c_1c_3+a_2^2+c_2^2)+2(a_4+c_4)-c_1^4]\\
&\quad - c_2[6a_1a_2-4a_1^3+3a_1c_1-2(a_3+c_3)] - c_3(3a_1^2-2a_2) + 2c_4a_1 - c_5 ,
\end{aligned}
\tag{B6}
$$

etc.

These relations are only of real interest, of course, when few coefficients have changed; otherwise it is better to begin from scratch. However, if, for instance, the first three values remain unchanged (i.e. for $k = 4$), the formulae simplify to

$$B_4 = b_4 - c_4 \quad \text{and}$$

$$B_5 = b_5 + 2a_1c_4 - c_5 ,$$

while the coefficients of lower order remain unchanged since $c_1 = c_2 = c_3 = 0$.

The application of this updating procedure to the evaluation of ${}_KT_1(N,N)$, when K is augmented, will be obvious and does not call for a numerical illustration.

C. Comparison with the previous approach

Some twenty years ago, a possible way for evaluating ${}_KT_1(N,N)$ was described in detail in [2]. The final result, which is also given in the review [1], may be written in the form

$${}_KT_1(N,N) = \frac{1 + x}{\Lambda_K} , \tag{C1}$$

with $\Lambda_K = \sum_{k=0}^{K} \lambda_k$.

The quantities λ_k are complicated functions of x, α and k, namely*

$$\lambda_k = \frac{e^{-s_k}}{k!} \{[1 + (1+\alpha)x - s_k]s_k^k + e^{\alpha x} s_{k+1}^{k+1}\} + (k+1)(1+\alpha x)\left[Q(k,s_{k+1}) - Q(k,s_k)\right] . \tag{C2}$$

In (C2) use has been made of the abbreviations

$$s_k = \max\{(1-k\alpha)x, 0\} \quad \text{and} \qquad Q(n,\mu) = \sum_{j=0}^{n} \frac{\mu^j}{j!} e^{-\mu} . \tag{C3}$$

This looks discouragingly complicated. Nevertheless, the numerical evaluation can be well mastered by a computer program and the main results thus obtained are presented in graphical form in [1].

Another problem, however, is the comparison of the expressions given above with the new formula (10): they seem to have little in common, at least at first sight. And yet, there can be no real doubt that they are fully equivalent, although it seems difficult to establish the identity in a formal way (see, however, Appendix D).

The complexity of the relations (C1) to (C3) has made it virtually impossible to derive, for example, approximate power-series expansions (of the type given in Table 1): as the developments were so long and tedious, the possibility that an error affecting the final result remained undetected would not be safely excluded. Indeed, independent repetitions of the calculations often led to slightly different results, and the origin of the discrepancy was sometimes difficult to locate. With the availability of the new approach described in the present report, this unpleasant situation could be brought to an end**.

A comparison of (C1) with (10) easily shows that identity of the two computational methods requires that

$$\Lambda_K = (1 + \alpha x)(1 + \Pi_K) . \tag{C4}$$

---

* We may note that the present quantities λ_k have been designated differently before, namely t_k' in [1] and ρt_k in [2].

** Another solution to this problem would have been to use one of the new computer-algebra systems, but none of them was available to us. Their possible introduction at BIPM is at present under discussion.

Before giving a general proof of the relation we first want to check its validity for some specific values of K. As this is a rather lengthy procedure, we shall only indicate briefly the route to follow and then limit ourselves to stating the results of the subsequent steps.

Let us begin with the case $\underline{K = 1}$.

Since now, with (C3),

$$s_o = x \ , \qquad s_1 = (1-\alpha)x \ , \qquad s_2 = 0$$

and

$$Q(0,s_o) = e^{-x} \ ,$$

$$Q(0,s_1) = e^{-(1-\alpha)x} \ ,$$

$$Q(1,s_1) = e^{-(1-\alpha)x}\left[1 + (1-\alpha)x\right] \ ,$$

$$Q(1,s_2) = 1 \ ,$$

we find from (C2), after some rearrangements,

$$\lambda_o = (1+x)\ e^{-(1-\alpha)x} \quad \text{and}$$

$$\lambda_1 = 2(1+\alpha x) - e^{-(1-\alpha)x}\left[2 + (1+\alpha)x\right] \ ,$$

thus

$$\Lambda_1 = (1+\alpha x)\left[2 - e^{-(1-\alpha)x}\right] \ . \tag{C5}$$

For $\underline{K = 2}$, the quantities s_o and s_1 are as before, whereas now $s_2 = (1-2\alpha)x$ and $s_3 = 0$.

Since

$$Q(1,s_1) = e^{-(1-\alpha)x}\left[1 + (1-\alpha)x\right] \ ,$$

$$Q(1,s_2) = e^{-(1-2\alpha)x}\left[1 + (1-2\alpha)x\right] \ ,$$

$$Q(2,s_2) = e^{-(1-2\alpha)x}\left[1 + (1-2\alpha)x + \frac{1}{2}(1-2\alpha)^2x^2\right] \ ,$$

$$Q(2,s_3) = 1 \ ,$$

we find

$$\lambda_o = (1+x)\ e^{-(1-\alpha)x} \ ,$$

$$\lambda_1 = -\ e^{-(1-\alpha)x}\left[2 + (1+\alpha)x\right] + e^{-(1-2\alpha)x}\left[2 + 2(1-\alpha)x + (1-2\alpha)x^2\right] \ ,$$

$$\lambda_2 = 3(1-\alpha x) - e^{-(1-2\alpha)x}\left[3 + 3(1-\alpha)x + (1-\alpha-2\alpha^2)x^2\right] \ ,$$

and hence for the sum

$$\Lambda_2 = 3(1+\alpha x) - e^{-(1-\alpha)x}(1+\alpha x) - e^{-(1-2\alpha)x}\left[1 + (1-\alpha)x + \alpha(1-2\alpha)x^2\right] . \tag{C6}$$

This procedure may now be continued, e.g. till K = 4, although this requires considerable patience.

After arrangement of the results to

$$\Lambda_2 = (1+\alpha x)\left\{3 - e^{-(1-\alpha)x} - e^{-(1-2\alpha)x}\left[1 + (1-2\alpha)x\right]\right\} ,$$

$$\Lambda_3 = (1+\alpha x)\left\{4 - e^{-(1-\alpha)x} - e^{-(1-2\alpha)x}\left[1 + (1-2\alpha)x\right] - e^{-(1-3\alpha)x}\left[1 + (1-3\alpha)x + \frac{1}{2}(1-3\alpha)^2x^2\right]\right\}$$

and (C7)

$$\Lambda_4 = (1+\alpha x)\left\{5 - e^{-(1-\alpha)x} - e^{-(1-2\alpha)x}\left[1 + (1-2\alpha)x\right] - e^{-(1-3\alpha)x}\left[1 + (1-3\alpha)x + \frac{1}{2}(1-3\alpha)^2x^2\right] - e^{-(1-4\alpha)x}\left[1 + (1-4\alpha)x + \frac{1}{2}(1-4\alpha)^2x^2 + \frac{1}{6}(1-4\alpha)^3x^3\right]\right\} ,$$

it becomes evident that all these expressions are in full agreement with (C4), since Π_K is given by (6). It may be worth mentioning that the last surviving errors in (C7) could only be eliminated once the simpler new approach was available. There can be little doubt that (C4) also holds beyond K = 4.

These controls clearly show the practical superiority of the new method. Without it, one could certainly not have thought, for instance, of deriving limiting relations for $K \gg 1$, as now described in section 6.

D. Proof of the equivalence

In view of the detailed results presented in Appendix C it seems more than likely that the two approaches available for the evaluation of ${}_K T_1(N,N)$ are equivalent, but a detailed formal proof of this statement would still be welcome. This is what we intend to achieve in this last part of the report.

Our starting point is obviously (C2) which we now write in the form

$$\lambda_k = \frac{1}{k!} e^{-s_k} \left[1 + (1+\alpha)x - (1-k\alpha)x\right] s_k^k + \frac{1}{k!} e^{-s_{k+1}} s_{k+1}^{k+1}$$
$$- (1+\alpha x) \left\{(k+1)\ Q(k,s_k) - (k+1)\ Q(k,s_{k+1})\right\} . \tag{D1}$$

Our aim is to arrive at a relation which is equal or equivalent to (C4) so that we have to sum up the different contributions λ_k. In view of (C3) it will be useful to single out the two special cases $s_{K+1} = 0$ and $Q(K, s_{K+1}) = 1$. For this reason we write

$$\Lambda_K = \sum_{k=0}^{K} \lambda_k$$
$$= \sum_{k=0}^{K} \frac{1}{k!} e^{-s_k} \left[1 + (k+1)\alpha x\right] s_k^k + \sum_{k=0}^{K-1} \frac{1}{k!} e^{-s_{k+1}} s_{k+1}^{k+1}$$
$$- (1+\alpha x) \left\{ \sum_{k=0}^{K} (k+1)\ Q(k,s_k) - \sum_{k=0}^{K-1} (k+1)\ Q(k,s_{k+1}) + K+1\right\}$$
$$= e^{-s_o}(1+\alpha x) + \sum_{k+1}^{K} e^{-s_k} \left\{\frac{1}{k!} (1+[k+1]\alpha x) + \frac{1}{(k-1)!}\right\} s_k^k$$
$$- (1+\alpha x) \left\{ \sum_{k=0}^{K} (k+1)\ Q(k,s_k) - \sum_{k=1}^{K} k\ Q(k-1,s_k)\right\} + (1+\alpha x)\ (K+1) . \tag{D2}$$

The expression in the curly brackets can be transformed to

$$\{\ldots\} = Q(0,s_o) + \sum_{k=1}^{K} (k+1)\ Q(k,s_k) - \sum_{k=1}^{K} k\ Q(k-1,s_k)$$
$$= e^{-x} + \sum_{k=1}^{K} Q(k,s_k) + \left[\sum_{k=1}^{K} k\ Q(k,s_k) - \sum_{k=1}^{K} k\ Q(k-1,s_k)\right] , \tag{D3}$$

we arrive for Λ_K at

$$\Lambda_K = (1+\alpha x)\ (K+1)\ e^{-x}(1+\alpha x) + \sum_{k=1}^{K} \frac{s_k^k}{k!}\, e^{-s_k} \left[(k+1) + (k+1)\alpha x\right]$$

$$- (1+\alpha x) \left[e^{-x} + \sum_{k=1}^{K} Q(k,s_k) + \sum_{k=1}^{K} \frac{s_k^k}{(k-1)!}\ e^{-s_k}\right]$$

$$= (1+\alpha x) \left[K+1 + \sum_{k=1}^{K} \frac{s_k^k}{k!}\, e^{-s_k}(k+1-k) - \sum_{k=1}^{K} Q(k,s_k)\right]$$

$$= (1+\alpha x) \left[K+1 + \sum_{k=1}^{K} \frac{s_k^k}{k!}\, e^{-s_k} - \sum_{k=1}^{K} \sum_{j=0}^{k} \frac{s_k^j}{j!}\, e^{-s_k}\right]$$

$$= (1+\alpha x) \left[K+1 - \sum_{k=1}^{K} \sum_{j=0}^{k-1} \frac{s_k^j}{j!}\, e^{-s_k}\right] . \qquad \text{(D4)}$$

By substitution of (C3) this can also be written as

$$\Lambda_K = (1+\alpha x) \left\{K+1 \sum_{k=1}^{K} \sum_{j=0}^{k-1} \frac{1}{j!} \left[(1-k\alpha)\ x\right]^j e^{-(1-k\alpha)x}\right\}$$

$$= (1+\alpha x) \left\{1 + \sum_{k=1}^{K} \left[1 - e^{-(1-k\alpha)x} \sum_{j=0}^{k-1} \frac{\left[(1-k\alpha)x\right]^j}{j!}\right]\right\} . \qquad \text{(D5)}$$

Remembering the definition of Π_K as given in (6), we are finally led to

$$\Lambda_K = (1 + \alpha x)\ (1 + \Pi_K) ,$$

which is just the condition required by (C4). This therefore concludes the formal proof that the two approaches for evaluating the transmission factor ${}_K T_1(N,\dot{N})$ are fully equivalent.

* * *

The author cannot help having a certain satisfaction in noting that, with the present report, he has succeeded in improving on one of his earliest contributions to the field of counting statistics - and this without the need to correct any errors. After all, this probably just confirms the well-known fact that a better approach nearly invariably leads to more insight, reliability and simplicity.

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