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# Decay effects with a generalized dead time*

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## Abstract

We describe the combined effect of decay and background on the observed count rate of a radioactive source, the pulses of which have passed a dead time of the generalized type. The formulae applicable to the two traditional types are recovered as special cases and their transition to the situation of negligible decay is also indicated.

## 1. Introduction

It is just a quarter of a century ago that the combined effects of decay, dead time (of traditional type) and background on the observed count rate of a radioactive source have been treated in a definite way. Indeed, the paper by Axton and Ryves [1] has left subsequent authors little to add, apart from studying higher moments, in particular the variances [2, 3], and this situation has hardly changed in the past few years. Likewise, by starting from the effect of decay on the Poisson distribution [4], most of the earlier results could be rederived and confirmed.

Recently the advent of dead times of a generalized type has brought some fresh air into the field. It is tempting, therefore, to see whether these new developments can be used for arriving at a unified general description which includes the two traditional types of dead times, previously treated separately, as special cases. This is what we try to achieve in the present report.

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* This paper is dedicated to Edward J. Axton on the occasion of his forthcoming 65th birthday. Though best known for his outstanding work in the field of neutron metrology, Ted has also shown a permanent and active interest in measurements of radioactivity.

## 2. The physical situation

As has been done in [1] we assume that there is a radioactive source decaying exponentially with a single decay constant $\lambda > 0$, to which a time-independent background rate $\rho_b$ is superimposed, hence

$$\rho(t) = \rho_o e^{-\lambda t} + \rho_b . \tag{1}$$

Here the origin t = 0 is an arbitrarily chosen reference time. It will be practical in what follows to choose as starting point the beginning of a measuring interval of duration T, thus for instance $t_o$ (Fig. 1).

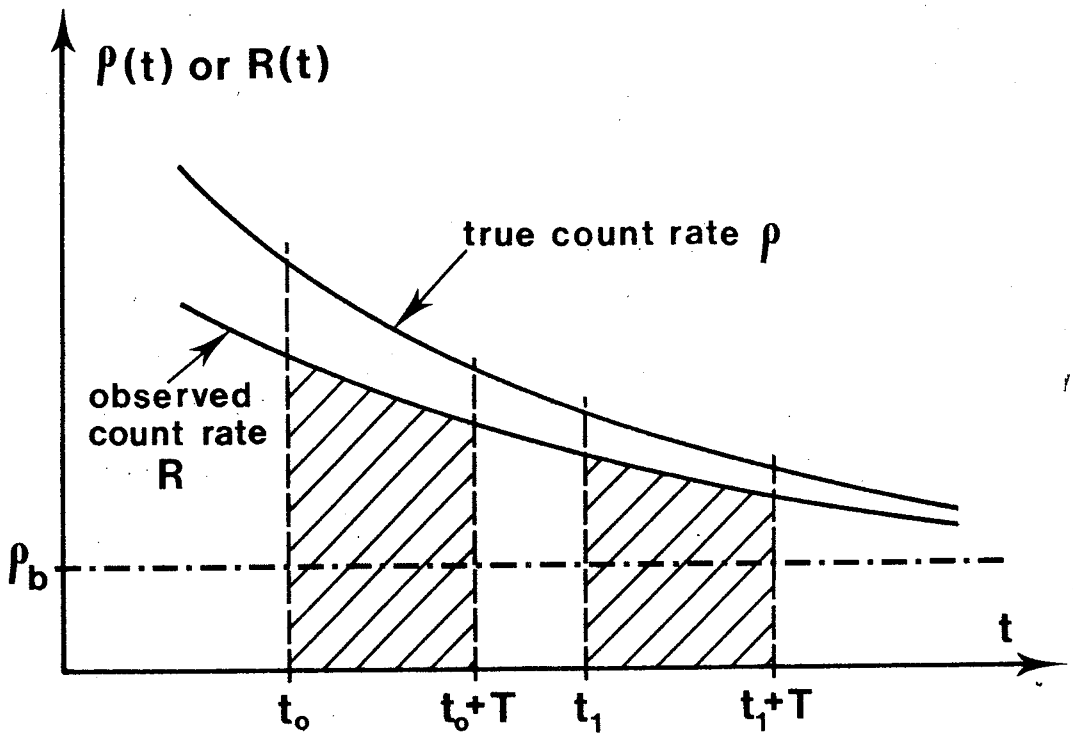


Fig. 1 - Schematic representation of true and observed count rates as a function of time. While ρ shows a simple exponential decay, R includes the dead-time losses. The hatched areas correspond to T $\bar{R}$ (cf. eq. (9)).

If we now assume that the counting losses of an original Poisson process are due to a generalized dead time of length $\tau$ and type $\theta$, then the original input rate $\rho(t)$ corresponds to an output rate given by [5]

$$_\theta R(t) = \frac{\theta \rho(t)}{e^{\theta\rho(t)\tau} + \theta - 1}$$

$$= \frac{\theta(\rho_o e^{-\lambda t} + \rho_b)}{\exp[\theta\tau(\rho_o e^{-\lambda t} + \rho_b)] + \theta - 1} \equiv \frac{\theta N}{\theta D} . \tag{2}$$

## 3. Towards the general solution

By the help of the abbreviations

$$\rho_o \tau = x_o \,, \qquad \rho_b \tau = x_b \qquad \text{and} \qquad x_o\, e^{-\lambda t} = \tilde{x} \tag{3}$$

the denominator of (2) can be developed into

$$\theta D = e^{\theta x_b} \left[1 + \theta \tilde{x}_o + \frac{1}{2}(\theta \tilde{x}_o)^2 + \frac{1}{6}(\theta \tilde{x}_o)^3 + \frac{1}{24}(\theta \tilde{x}_o)^4 + \dots\right] + \theta - 1$$

$$= (e^{\theta x_b} - 1) + \theta\Big[1 + e^{\theta x_b}\tilde{x} + e^{\theta x_b}\frac{1}{2}\theta \tilde{x}^2$$

$$+ e^{\theta x_b}\frac{1}{6}\theta^2 \tilde{x}^3 + e^{\theta x_b}\frac{1}{24}\theta^3 \tilde{x}^4 + \dots\Big] .$$

By putting*

$$1 + \frac{e^{\theta x_b} - 1}{\theta} = \frac{1}{\alpha} \qquad \text{and} \qquad \alpha\, e^{\theta x_b} = \beta \tag{4}$$

we can write

$$D = \frac{1}{\alpha}\left[1 + \beta \tilde{x} + \frac{1}{2}\beta\theta \tilde{x}^2 + \frac{1}{6}\beta\theta^2 \tilde{x}^3 + \frac{1}{24}\beta\theta^3 \tilde{x}^4 + \dots\right]$$

$$= \frac{1}{\alpha}\left[1 + \frac{\beta}{\theta}(e^{\theta\tilde{x}} - 1)\right] . \tag{5}$$

The corresponding series development

$$D = \frac{1}{\alpha}\left(1 + \sum_{k=1}^{\infty} a_k \tilde{x}^k\right) ,$$

with $a_k = \dfrac{\beta}{k!}\, \theta^{k-1}$ ,

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* Some readers might like to have the respective series expansions at hand which are

$$\alpha = 1 - x_b\left[1 - (1 - \frac{1}{2}\theta)x_b + (1 - \theta + \frac{1}{6}\theta^2)x_b^2 \pm \dots\right] \quad \text{and}$$

$$\beta = 1 - (1-\theta)x_b\left[1 - (1 - \frac{1}{2}\theta)x_b + (1 - \theta + \frac{1}{6}\theta^2)x_b^2 \mp \dots\right] . \tag{4'}$$

can be inverted to yield

$$D^{-1} = \alpha\,(1 + \sum_{k=1}^{\infty} b_k\,\tilde{x}^k)\ , \tag{6}$$

where the first new coefficients can be found by means of [6] as

$$\begin{aligned} b_1 &= -a_1 = -\beta\ , \\ b_2 &= a_1^2 - a_2 = \beta^2 - \frac{1}{2}\beta\theta\ , \\ b_3 &= -a_1^3 + 2a_1a_2 - a_3 = -\beta^3 + \beta^2\theta - \frac{1}{6}\beta\theta^2\ , \\ b_4 &= a_1^4 - 3a_1a_2 + 2a_1a_3 + a_2^2 - a_4 \\ &= \beta^4 - \frac{3}{2}\beta^3\theta + \frac{7}{12}\beta^2\theta^2 - \frac{1}{24}\beta\theta^3\ . \end{aligned} \tag{7}$$

Therefore

$$\begin{aligned} {}_\theta R(t) &= (\rho_o\,e^{-\lambda t} + \rho_b)\,\alpha\,[1 - \beta\tilde{x} + \beta(1 - \frac{1}{2}\theta)\,\tilde{x}^2 \mp \ldots] \\ &= \frac{\alpha}{\tau}(\tilde{x} + x_b)\,[1 - \beta\tilde{x} + \beta(\beta - \frac{1}{2}\theta)\tilde{x}^2 - \beta(\beta^2 - \beta\theta + \frac{1}{6}\theta^2)\tilde{x}^3 \\ &\qquad + \beta(\beta^3 - \frac{3}{2}\beta^2\theta + \frac{7}{12}\beta\theta^2 - \frac{1}{24}\theta^3)\tilde{x}^4 \mp \ldots]\ . \end{aligned} \tag{8}$$

We now want to evaluate the mean count rate ${}_\theta\bar{R}$ observed during the measuring time T, thus

$${}_\theta\bar{R} = \frac{1}{T}\int_o^T {}_\theta R(t)\,dt\ . \tag{9}$$

A look at (8) shows that this leads to integrals of the type

$$\int_o^T \tilde{x}^k\,dt = x_o^k \int_o^T e^{-k\lambda t}\,dt = \frac{x_o^k}{k\lambda}(1 - e^{-k\lambda T})\ , \quad \text{with } k \geqslant 1\ . \tag{10}$$

We therefore find (with $\kappa = \lambda T$)

$$\theta^{\bar{R}} = \frac{\alpha x_b}{T\tau} \Big[T - \beta \frac{x_o}{\lambda} (1-e^{-\kappa}) + \beta(\beta - \frac{1}{2}\theta) \frac{x_o^2}{2\lambda} (1-e^{-2\kappa})$$

$$- \beta(\beta^2 - \beta\theta + \frac{1}{6}\theta^2) \frac{x_o^3}{3\lambda} (1-e^{-3\kappa})$$

$$+ \beta(\beta^3 - \frac{3}{2}\beta^2\theta + \frac{7}{12}\beta\theta^2 - \frac{1}{24}\theta^3) \frac{x_o^4}{4\lambda} (1-e^{-4\kappa}) \mp \ldots\Big]$$

$$+ \frac{\alpha}{T\tau} \Big[\frac{x_o}{\lambda} (1-e^{-\kappa}) - \beta \frac{x_o^2}{2\lambda}(1-e^{-2\kappa}) + \beta(\beta - \frac{1}{2}\theta) \frac{x_o^3}{3\lambda} (1-e^{-3\kappa})$$

$$- \beta(\beta^2 - \beta\theta + \frac{1}{6}\theta^2) \frac{x_o^4}{4\lambda} (1-e^{-4\kappa})$$

$$+ \beta(\beta^3 - \frac{3}{2}\beta^2\theta + \frac{7}{12}\beta\theta^2 - \frac{1}{24}\theta^3) \frac{x_o^5}{5\lambda} (1-e^{-5\kappa}) \mp \ldots\Big]$$

$$= \alpha\rho_b \Big[1 - \beta \frac{x_o}{\kappa} (1-e^{-\kappa}) + \beta(\beta - \frac{1}{2}\theta) \frac{x_o^2}{2\kappa}(1-e^{-2\kappa}) \mp \ldots\Big]$$

$$+ \alpha\rho_o \Big[\frac{1}{\kappa}(1-e^{-\kappa}) - \beta \frac{x_o}{2\kappa}(1-e^{-2\kappa}) + \beta(\beta - \frac{1}{2}\theta) \frac{x_o^2}{3\kappa}(1-e^{-3\kappa}) \mp \ldots\Big] .$$

With the abbreviation $e^{-\kappa} = \Lambda$ we arrive at

$$\theta^{\bar{R}} = \frac{\alpha\rho_b}{\kappa} \Big[\kappa - \beta(1-\Lambda)\, x_o + \frac{1}{2}\beta(\beta - \frac{1}{2}\theta)\, (1-\Lambda^2)\, x_o^2 \mp \ldots\Big]$$

$$+ \frac{\alpha\rho_o}{\kappa} \Big[1-\Lambda - \frac{1}{2}\beta(1-\Lambda^2)\, x_o + \frac{1}{3}\beta(\beta - \frac{1}{2}\theta)\, (1-\Lambda^3)\, x_o^2 \mp \ldots\Big]. \qquad (11)$$

By a closer look at the factors appearing in (11) which involve the parameter $\theta$ we may realize that they bear a close resemblance to expressions we met previously. In particular, it has been found in [5] that the output R of a generalized dead time $(\tau, \theta)$ can be written (for a Poissonian input rate $\rho$) as

$$R = \rho \Big[1 + \sum_{n=1}^{\infty} B_n\, (-\rho\tau)^n\Big] , \qquad (12)$$

with $B_n = \frac{1}{n!} \sum_{k=1}^{n} k!\; S(n,k)\; (-\theta)^{n-k}$ , where S(n,k) are the Stirling numbers of the second kind, extensively listed e.g. in [7]. The first few coefficients in (12) turn out to be

$$\begin{aligned} B_1 &= 1 , \\ B_2 &= 1 - \frac{1}{2}\,\theta , \\ B_3 &= 1 - \theta + \frac{1}{6}\,\theta^2 , \\ B_4 &= 1 - \frac{3}{2}\,\theta + \frac{7}{12}\,\theta^2 - \frac{1}{24}\,\theta^3, \quad \text{etc.} \end{aligned} \tag{13}$$

A detailed comparison of the terms appearing in the expansions (11) and (12) allows us to arrive at the general form valid for the series development of $\theta\bar{R}$

$$\theta\bar{R} = \frac{\alpha\rho_b}{\kappa}\,[\kappa + \sum_{n=1}^{\infty} (\frac{1-\Lambda^n}{n})\,(-x_o)^n \sum_{k=1}^{n} \frac{k!}{n!}\,S(n,k)\,(-\theta)^{n-k}\,\beta^k]$$

$$+ \frac{\alpha\rho_o}{\kappa}\,[1-\Lambda + \sum_{n=1}^{\infty} (\frac{1-\Lambda^{n+1}}{n+1})\,(-x_o)^n \sum_{k=1}^{n} \frac{k!}{n!}\,S(n,k)\,(-\theta)^{n-k}\,\beta^k] ,$$

or also

$$\theta\bar{R} = \alpha\,[\frac{\rho_o}{\kappa}\,(1-\Lambda) + \rho_b]$$

$$+ \frac{\alpha}{\kappa}\,\{\sum_{n=1}^{\infty} \frac{(-x_o)^n}{n!}\,[\rho_o\,(\frac{1-\Lambda^{n+1}}{n+1}) + \rho_b\,(\frac{1-\Lambda^n}{n})] \sum_{k=1}^{n} k!\,S(n,k)\,(-\theta)^{n-k}\,\beta^k\} . \tag{14}$$

This rather complicated expression is the main outcome of the present study. As it is supposed to be a generalization of the results contained in [1], one should be capable of recovering them. This will be done in sections 5 and 6.

In addition, it should be possible, of course, to go back to the case of negligible decay during the measuring period. However, simply putting $\kappa = 0$ in (14) would lead to ill-defined expressions. Since the limiting process is not entirely trivial, it might be worthwhile to perform it for checking purposes.

## 4. The transition to $\lambda = 0$

A look at (14) shows that the required transition first calls for the evaluation of

$$\lim_{\kappa\to 0}\,[\frac{1-\Lambda^n}{\kappa}] = \lim_{\kappa\to 0}\,[\frac{1}{\kappa}\,(n\kappa - \frac{1}{2}\,n^2\kappa^2 \pm \ldots)] = n . \tag{15}$$

This then leads to

$$ {}_{\theta}\overline{R}\tau = \alpha\tau(\rho_o+\rho_b) + \alpha\tau \left[ \sum_{n=1}^{\infty} \frac{(-x_o)^n}{n!} (\rho_o+\rho_b) \sum_{k=1}^{n} k!\ S(n,k)\ (-\theta)^{n-k}\beta^k \right] $$

$$ = \alpha(x_o+x_b) \left[1 + \sum_{n=1}^{\infty} c_n\ (-\beta x_o)^n\right] $$

$$ = \frac{\alpha(x_o+x_b)}{\beta\rho_o} \{\beta\rho_o[1 + \sum_{n=1}^{\infty} c_n\ (-\beta x_o)^n]\} , \tag{16} $$

with

$$ c_n = \frac{1}{n!} \sum_{k=1}^{n} k!\ S(n,k)\ (-\theta/\beta)^{n-k} . $$

A comparison with (12) reveals that the expression in the curly brackets is just the output count rate of a generalized dead time, provided that we identify $\rho_o$ with $\beta\rho_o$ and $\theta$ with $\theta/\beta$. Hence, by means of the Takács formula [5] we can now write

$$ {}_{\theta}\overline{R}\tau = \frac{\alpha(x_o+x_b)}{\beta\rho_o} \left[\frac{(\theta/\beta)\beta\rho_o}{e^{(\theta/\beta)\beta x_o} + \theta/\beta - 1}\right] = \alpha(x_o+x_b) \frac{\theta}{\beta\, e^{\theta x_o} + \theta - \beta} . $$

Since by virtue of (4) we have

$$ \frac{\beta(e^{\theta x_o}-1)}{\alpha} + \frac{\theta}{\alpha} = e^{\theta x_b} (e^{\theta x_o}-1) + \theta + e^{\theta x_b} - 1 = e^{\theta(x_o+x_b)} + \theta - 1 , $$

one finally arrives at the expected result

$$ {}_{\theta}\overline{R}\tau = \frac{\theta(x_o + x_b)}{e^{\theta(x_o+x_b)} + \theta - 1} , \tag{17} $$

where $\rho_o + \rho_b$ is the (constant) total input count rate.

## 5. The case of a non-extended dead time

For $\theta \to 0$ it follows from (4) that

$$ \alpha = \beta = \frac{1}{1 + x_b} \tag{18} $$

and also that in (14) the summation over k has only the single contribution from k = n. Therefore, since S(n,n) = 1,

$$ {}_{o}\overline{R} = \frac{\alpha\rho_o}{\kappa} (1-\Lambda) + \alpha\rho_b + \frac{\alpha}{\kappa} \{ \sum_{n=1}^{\infty} (-x_o)^n\ \beta^n\ [\rho_o\ (\frac{1-\Lambda^{n+1}}{n+1}) + \rho_b\ (\frac{1-\Lambda^n}{n})]\} , $$

or likewise

$$
{}_o\overline{R} = \frac{\alpha\rho_o}{\kappa}(1-\Lambda) + \alpha\rho_b + \frac{\alpha\rho_o}{\kappa}\,{}_oA + \frac{\alpha\rho_b}{\kappa}\,{}_oB ,
$$

$$
\text{with}\quad {}_oA = \sum_{n=1}^{\infty} \frac{(-\beta x_o)^n}{n+1}(1-\Lambda^{n+1}) \quad \text{and}
$$

$$
{}_oB = \sum_{n=1}^{\infty} \frac{(-\beta x_o)^n}{n}(1-\Lambda^{n}) . \tag{19}
$$

The evaluation of these sums first yields

$$
{}_oA = \frac{-1}{\beta x_o}\sum_{n=2}^{\infty}\frac{(-\beta x_o)^n}{n}(1-\Lambda^n)
$$

$$
= \frac{-1}{\beta x_o}\left[\beta x_o + \sum_{n=1}^{\infty}\frac{(-\beta x_o)^n}{n}\right] + \frac{1}{\beta x_o}\left[\Lambda\beta x_o + \sum_{n=1}^{\infty}\frac{(-\Lambda\beta x_o)^n}{n}\right] .
$$

Since $\ln(1+z) = -\sum_{n=1}^{\infty}\frac{(-z)^n}{n}$ , for $z^2 < 1$ ,

we arrive at

$$
{}_oA = -1 + \Lambda + \frac{1}{\beta x_o}\ln(1+\beta x_o) - \frac{1}{\beta x_o}\ln(1+\Lambda\beta x_o) , \tag{20}
$$

and likewise

$$
{}_oB = \sum_{n=1}^{\infty}\frac{(-\beta x_o)^n}{n} - \sum_{n=1}^{\infty}\frac{(-\Lambda\beta x_o)^n}{n} = \ln\left[\frac{1+\Lambda\beta x_o}{1+\beta x_o}\right] . \tag{21}
$$

Hence (18) becomes

$$
{}_o\overline{R} = \frac{\alpha\rho_o}{\kappa}(1-\Lambda) + \alpha\rho_b + \frac{\alpha\rho_o}{\kappa}\left\{-1+\Lambda+\frac{1}{\beta x_o}\ln\left[\frac{1+\beta x_o}{1+\Lambda\beta x_o}\right]\right\} - \frac{\alpha\rho_b}{\kappa}\ln\left[\frac{1+\beta x_o}{1+\Lambda\beta x_o}\right]
$$

$$
= \alpha\rho_b + \ln\left[\frac{1+\beta x_o}{1+\Lambda\beta x_o}\right]\left\{\frac{\alpha\rho_o}{\kappa}\frac{1}{\beta x_o} - \frac{\alpha\rho_b}{\kappa}\right\}
$$

or also, since

$$
\frac{\alpha\rho_o}{\kappa}\frac{1}{\beta x_o} - \frac{\alpha\rho_b}{\kappa} = \frac{1}{\kappa}\left(\frac{1}{\tau} - \alpha\rho_b\right) = \frac{1}{\kappa\tau}(1-\alpha x_b) = \frac{1}{\kappa\tau}\frac{1}{1+x_b} ,
$$

$$
{}_o\overline{R} = \alpha\rho_b + \frac{1}{\kappa\tau}\frac{1}{1+x_b}\ln\left[\frac{1+\beta x_o}{1+\Lambda\beta x_o}\right] .
$$

When expressed in the original variables this corresponds to

$$ {}_{o}\bar{R} = \frac{\rho_b}{1 + x_b} + \frac{1}{\lambda T\tau(1 + x_b)} \ln \left[\frac{1 + x_b + x_o}{1 + x_b + x_o e^{-\lambda T}}\right] . \qquad (22) $$

This result agrees with eq. (2) in [1].

6. The case of an extended dead time

For $\theta = 1$ we readily see from (4) that

$$ \alpha = e^{-x_b} \quad \text{and} \quad \beta = 1 . \qquad (23) $$

Hence (14) becomes

$$ {}_{1}\bar{R} = \alpha \left[\frac{\rho_o}{\kappa}(1-\Lambda) + \rho_b\right] + \frac{\alpha}{\kappa} \sum_{n=1}^{\infty} \frac{(-x_o)^n}{n!} \left[\rho_o \left(\frac{1-\Lambda^{n+1}}{n+1}\right) + \rho_b \left(\frac{1-\Lambda^n}{n}\right)\right] , $$

since the Stirling numbers have the property [7] that

$$ \sum_{k=1}^{n} k!\, S(n,k)\, (-1)^{n-k} = 1 . \qquad (24) $$

Thus ${}_{1}\bar{R}$ can also be expressed by

$$ {}_{1}\bar{R} = \alpha \left[\frac{\rho_o}{\kappa}(1-\Lambda) + \rho_b\right] + \frac{\alpha\rho_o}{\kappa}\, {}_{1}A + \frac{\alpha\rho_b}{\kappa}\, {}_{1}B , $$

$$ \text{with} \quad {}_{1}A = \sum_{n=1}^{\infty} \frac{(-x_o)^n}{n!} \left(\frac{1 - \Lambda^{n+1}}{n+1}\right) \quad \text{and} \qquad (25) $$

$$ {}_{1}B = \sum_{n=1}^{\infty} \frac{(-x_o)^n}{n!} \left(\frac{1 - \Lambda^n}{n}\right) . $$

The evaluation of the first sum gives

$$ \begin{aligned} {}_{1}A &= \frac{-1}{x_o} \left[\sum_{n=1}^{\infty} \frac{(-x_o)^n}{n!} + x_o\right] + \frac{1}{x_o} \left[\sum_{n=1}^{\infty} \frac{(-\Lambda x_o)^n}{n!} + \Lambda x_o\right] \\ &= -1 - \frac{1}{x_o}(e^{-x_o} - 1) + \Lambda + \frac{1}{x_o}(e^{-\Lambda x_o} - 1) \\ &= \Lambda - 1 - \frac{1}{x_o}(e^{-x_o} - e^{-\Lambda x_o}) . \qquad (26) \end{aligned} $$

The second sum

$$ {}_1B = \sum_{n=1}^{\infty} \frac{(-x_o)^n}{n\,n!} - \sum_{n=1}^{\infty} \frac{(-\Lambda x_o)^n}{n\,n!} $$

will remind us of the series expansion of the exponential integral which is [7]

$$ E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} \frac{(-z)^n}{n\,n!} \ . $$

Therefore

$$ \begin{aligned} {}_1B &= -\ln x_o - E_1(x_o) + \ln(\Lambda x_o) + E_1(\Lambda x_o) \\ &= -\kappa + E_1(\Lambda x_o) - E_1(x_o) \ , \end{aligned} \qquad (27) $$

since $\Lambda = e^{-\kappa}$.

We are thus led to

$$ \begin{aligned} {}_1\overline{R} &= \alpha \left[\frac{\rho_o}{\kappa}(1-\Lambda) + \rho_b)\right] - \frac{\alpha\rho_o}{\kappa}\left[1-\Lambda + \frac{1}{x_o}\left(e^{-x_o} - e^{-\Lambda x_o}\right)\right] \\ &\quad + \frac{\alpha\rho_b}{\kappa}\left[-\kappa + E_1(\Lambda x_o) - E_1(x_o)\right] \\ &= \frac{\alpha}{\kappa}\left\{\frac{\rho_o}{x_o}\left(e^{-\Lambda x_o} - e^{-x_o}\right) + \rho_b\left[E_1(\Lambda x_o) - E_1(x_o)\right]\right\} \ . \end{aligned} $$

In the original variables (except for $\Lambda = e^{-\lambda T}$) this corresponds to

$$ {}_1\overline{R} = \frac{e^{-x_b}}{\lambda T \tau} \left\{e^{-\Lambda x_o} - e^{-x_o} + x_b \left[E_1(\Lambda x_o) - E_1(x_o)\right]\right\} , \qquad (28) $$

which is in agreement with eq. (5) in [1].

It may be mentioned that several years ago - in thoughtless neglect of ref. [1] - the relevant expressions for ${}_o\overline{R}$ and ${}_1\overline{R}$ have been rederived by starting from a decay-distorted Poisson law. While the formulae (expressed as expectation values ${}_\lambda E(k) = \overline{R}\, t_o$) turned out to be quite simple in the case of negligible background [4], they became rather involved for $\rho_b \neq 0$. Nevertheless, the general results have been evaluated for a non-extended [8] as well as for an extended dead time [9] and it can be verified that they agree with (22) and (28). It must be admitted, however, that the choice of the auxiliary quantities in [8] and [9] was not a very lucky one; use of the new abbreviations makes the expressions look much simpler, although in fact they are completely equivalent.

Looking back to the general formula (14) one might think that it is too difficult for a practical application when $\theta$ has an intermediate value. However, this is not the case since (14) can be readily programmed and numerically evaluated to any order n of terms necessary for the precision required. As for the Stirling numbers, their determination is easily obtained by means of the recursion formula (for $n \geq 2$ and $1 \leq k \leq n$)

$$S(n,k) = k\, S(n-1,k) + S(n-1,k-1) , \tag{29}$$

using for the start, for instance, the fact that the limiting values are $S(n,1) = S(n,n) = 1$.

## 7. Concluding remarks

An important practical aspect of the considerations treated above has not yet been touched upon, namely the reversion problem. This arises in a natural way in that for most problems it is the initial count rate $\rho_o$ that we wish to know on the basis of the measured (average) rate $\overline{R}$. This is just the reverse order of what we have done till now.

The absence of this subject from what has been presented above is not an oversight, but is due to our present ignorance. Indeed, the only case, where the reversion is well known concerns the arrangement with a non-extended dead time ($\theta = 0$), and this solution has already been given in Axton and Ryves' classic paper [1]. The result is (in our present notation and with some brackets added)

$$\rho_o = \left(\frac{1 + x_b}{\tau}\right) \left\{ \frac{\exp\,[(1-y)(1+x_b)\kappa] - \exp\,(\kappa)}{1 - \exp\,[(1-y)(1+x_b)\kappa]} \right\} , \tag{30}$$

$$\text{with} \quad y = {}_o\overline{R}\tau \quad \text{and} \quad x_b = \frac{\overline{\rho}_b \tau}{1 - \overline{\rho}_b \tau} ,$$

where $\overline{\rho}_b$ is the observed background rate.

In the absence of background, (30) can be simplified to

$$\rho_o \tau = x_o = \frac{e^{\kappa y} - 1}{1 - e^{-\kappa(1-y)}} . \tag{31}$$

A derivation of (30) is sketched in the Appendix.

For the general case with $0 < \theta \leq 1$ there are, in principle, always two solutions for the original count rate $\rho_o$ which correspond to a measured value $\overline{R}$, but for the time being no general formulae are known for their analytic evaluation. In practice, therefore, numerical methods have to be used for their determination by means of iterations. However, in view of recent progress with similar reversion problems, in particular by Libert [10], there is good reason to hope for a solution of the general case, perhaps in the near future.

For the frequent case where decay effects are quite small, i.e. $\kappa = \lambda T \ll 1$, most of the formulae given in this report will not be very practical to apply since they become ill-defined for $\kappa = 0$. For tackling this situation a series development in terms of $\kappa$ may be more appropriate.

As an example we consider the situation for a non-extended dead time and no background. If we put, as usual,

$$ {}_{o}\bar{R} = T_o \, \rho_o \, , $$

then the transmission factor $T_o$, expressed in terms of the measured quantity $y = {}_{o}\bar{R}\tau$, can be shown to be given by the approximation

$$ T_o \cong (1-y) \left[1 - \frac{\kappa}{2} + \frac{\kappa^2}{12}(2-y) - \frac{\kappa^3}{24}(1-y) + \frac{\kappa^4}{720}(6-9y+y^2+y^3) - \frac{\kappa^5}{1\,440}(2-4y+y^2+y^3)\right] . $$

A number of similar relations could be obtained, but we do not wish to go here into more details. The intrinsic flexibility of mathematical descriptions ensures us that an appropriate form can always be found.

APPENDIX

Some complements

Obviously, the detailed calculations to be presented in this appendix are not new for the authors of [1], but they may be welcome to those who cannot afford to spend the time and effort necessary for evaluating in detail the results presented in [1]. After all, most users of formulae feel more at ease if they have the possibility to follow step by step their derivation instead of accepting them in good faith.

Let us therefore give a concise derivation of eqs. (2), (5) and (2a) in [1], using our present notation.

a) Non-extended dead time

For a source decaying in time according to (1), the average count rate ${}_{o}\bar{R}$, measured during a time interval T after a non-extended dead time $\tau$, is given by

$$ {}_{o}\bar{R} = \frac{1}{T} \int_0^T (\rho_o \, e^{-\lambda t} + \rho_b) \, [1 + (\rho_o \, e^{-\lambda t} + \rho_b)\tau]^{-1} \, dt \, . \qquad \text{(A1)} $$

For the new variable $z = e^{-\lambda t}$ and with the abbreviations $x_o = \rho_o \tau$, $x_b = \rho_b \tau$ and $\Lambda = e^{-\lambda T}$ we arrive at

$$ {}_o\bar{R}\, T = \int_\Lambda^1 \frac{\rho_o z + \rho_b}{1 + x_o z + x_b} \frac{dz}{\lambda z} $$

$$ = \frac{x_o}{\lambda\tau} \int_\Lambda^1 \frac{dz}{1 + x_o z + x_b} + \frac{x_b}{\lambda\tau} \int_\Lambda^1 \frac{dz}{z(1 + x_o z + x_b)} . \qquad \text{(A2)} $$

The integrals are of the well-known types

$$ \int \frac{dz}{a + bz} = \frac{1}{b} \ln |a + bz| \quad \text{and} $$

$$ \int \frac{dz}{z(a + bz)} = \frac{-1}{a} \ln \left|\frac{a + bz}{z}\right| . \qquad \text{(A3)} $$

Therefore (with $a = 1 + x_b$ and $b = x_o$)

$$ {}_o\bar{R}\, T = \frac{1}{\lambda\tau} \ln \left[\frac{1 + x_b + x_o}{1 + x_b + \Lambda x_o}\right] - \frac{x_b}{\lambda\tau(1+x_b)} \ln \left[\frac{1 + x_b + x_o}{1 + x_b + \Lambda x_o} \Lambda\right] $$

or also, since $\ln \Lambda = -\lambda T$,

$$ {}_o\bar{R}\, \tau = \frac{x_b}{1 + x_b} + \ln \left[\frac{1 + x_b + x_o}{1 + x_b + \Lambda x_o}\right] \left[\frac{1}{\lambda T} - \frac{x_b}{\lambda T(1+x_b)}\right] . $$

This gives for the average count rate

$$ {}_o\bar{R} = \frac{1}{1+x_b} \left\{\rho_b + \frac{1}{\lambda T \tau} \ln \left[\frac{1 + x_b + x_o}{1 + x_b + \Lambda x_o}\right]\right\} , \qquad \text{(A4)} $$

in agreement with eq. (2) in [1], for $\lambda > 0$.

As we have seen in the general case, the transition to negligible decay cannot be obtained simply by putting $\lambda = 0$ in (A4). The limiting process requires the evaluation of ($\kappa = \kappa T$)

$$ J \equiv \lim_{\kappa \to 0} \left\{ \frac{1}{\kappa} \ln \left[\frac{1 + x_b + x_o}{1 + x_b + \Lambda x_o}\right]\right\} . \qquad \text{(A5)} $$

Since $\quad \Lambda x_o = x_o\, e^{-\kappa} = x_o - \kappa x_o(1 - \frac{1}{2} \kappa \pm \ldots)$ ,

and by using the general formula

$$ \ln (x+a) = \ln x + 2 \left[\frac{a}{2x+a} + \frac{a^3}{3(2x+a)^3} + \ldots\right] , $$

valid for $a^2 < (2x+a)^2$ ,

we see that (with $x = 1+x_b+x_o$ and $a \cong -\kappa x_o$)

$$\ln(1+x_b+\Lambda x_o) \cong \ln(1+x_b+x_o) - \frac{2\kappa x_o}{2(1+x_b+x_o) - \kappa x_o},$$

and therefore

$$J = \frac{x_o}{1+x_b+x_o}, \qquad \text{for } \kappa \to 0. \tag{A6}$$

Hence

$${}_o\bar{R}\tau = \frac{1}{1+x_b}\left[x_b + \frac{x_o}{1+x_b+x_o}\right],$$

which can be rearranged to yield the expected result

$${}_o\bar{R}\tau = \frac{x_o + x_b}{1 + x_o + x_b}. \tag{A7}$$

b) Extended dead time

For an extended dead time $\tau$, the corresponding evaluation of ${}_1\bar{R}$ can be done in the following way. We begin with

$${}_1\bar{R} = \frac{1}{T}\int_o^T (\rho_o\, e^{-\lambda t} + \rho_b) \exp\left[-(\rho_o\, e^{-\lambda t} + \rho_b)\tau\right] dt \tag{A8}$$

or, with the usual abbreviations,

$$\begin{aligned} {}_1\bar{R}\,T &= \rho_o\, e^{-x_b}\int_o^T e^{-\lambda t}\exp(-x_o\, e^{-\lambda t})\, dt \\ &\quad + \rho_b\, e^{-x_b}\int_o^T \exp(-x_o\, e^{-\lambda t})\, dt \\ &\equiv \rho_o\, e^{-x_b} I_1 + \rho_b\, e^{-x_b} I_2 . \end{aligned} \tag{A9}$$

By putting $e^{-\lambda t} = u$ and with $\Lambda = e^{-\lambda T}$ we obtain

$$I_1 = \int_o^T \exp(-\lambda t - x_o\, e^{-\lambda t})\, dt = \int_1^{e^{-\lambda T}} z\, e^{-x_o u}\frac{du}{(-\lambda u)} = \frac{1}{\lambda}\int_\Lambda^1 e^{-x_o u}\, du$$

$$= \frac{1}{\lambda x_o}\left(e^{-\Lambda x_o} - e^{-x_o}\right),$$

and likewise (with $x_o u = v$)

$$I_2 = \int_o^T \exp(-x_o e^{-\lambda t})\, dt = \int_1^{\Lambda} e^{-x_o u} \frac{du}{(-\lambda u)}$$

$$= \frac{1}{\lambda} \int_{\Lambda x_o}^{x_o} \frac{e^{-v}}{v} dv .$$

Since the exponential integral is defined by

$$E_1(z) = \int_1^{\infty} \frac{e^{-zt}}{t} dt = \int_z^{\infty} \frac{e^{-t}}{t} dt , \qquad \text{(A10)}$$

we can also write

$$I_2 = \frac{1}{\lambda} \left[ \int_{\Lambda x_o}^{\infty} \frac{e^{-v}}{v} dv - \int_{x_o}^{\infty} \frac{e^{-v}}{v} dv \right] = \frac{1}{\lambda} \left[ E_1(\Lambda x_o) - E_1(x_o) \right] .$$

Hence, the final result is

$$_1\bar{R} = \frac{e^{-x_b}}{\lambda T \tau} \{ e^{-\Lambda x_o} - e^{-x_o} + x_b [E_1(\Lambda x_o) - E_1(x_o)] \} , \qquad \text{(A11)}$$

in agreement with eq. (5) in [1], for $\lambda > 0$.

Again the step backwards from (A11) to negligible decay requires some attention. The first limiting process, however, is quite elementary. Since

$$e^{-\Lambda x_o} - e^{-x_o} = e^{-x_o} [e^{(1-\Lambda)x_o} - 1]$$

$$= e^{-x_o} [(1-\Lambda)\, x_o + \frac{1}{2} (1-\Lambda)^2\, x_o^2 \mp \ldots]$$

and $\frac{1-\Lambda}{\kappa} \to 1$ , for $\kappa \to 0$ ,

we have

$$\lim_{\kappa \to 0} [\frac{1}{\kappa} (e^{-\Lambda x_o} - e^{-x_o})] = x_o e^{-x_o} . \qquad \text{(A12)}$$

For the second part probably we had best use the relation

$$\frac{dE_1(z)}{dz} = -\frac{1}{z} e^{-z} .$$

Since

$$E_1(\Lambda x_o) = E_1(e^{-\kappa} x_o) \cong E_1(x_o - \kappa x_o) ,$$

we find that (for $\kappa \to 0$ and with $a = -\kappa x_o$)

$$\frac{E_1(\Lambda x_o) - E_1(x_o)}{\kappa} \cong \frac{E_1(x_o+a) - E_1(x_o)}{a/(-x_o)} \cong \frac{dE_1(x_o)}{dx_o}(-x_o) = e^{-x_o} . \quad \text{(A13)}$$

Hence, it follows from (A11) for $\lambda \to 0$ that

$$_1\overline{R}\tau = e^{-x_b} \{x_o e^{-x_o} + x_b e^{-x_o}\} = (x_o+x_b) e^{-(x_o+x_b)} , \quad \text{(A14)}$$

as expected.

In the absence of a background contribution (i.e. for $x_b = 0$) the results (A4) and (A11) can be simplified, yielding then

$$_o\overline{R} = \frac{1}{\lambda T \tau} [\ln (1 + x_o) - \ln (1 + \Lambda x_o)] \quad \text{(A15)}$$

and

$$_1\overline{R} = \frac{1}{\lambda T \tau} [e^{-\Lambda x_o} - e^{-x_o}] . \quad \text{(A16)}$$

Note that eq. (5a) in [1], which corresponds to (A16), has been distorted by some obvious misprints.

### c) Original count rate

Our final supplement concerns eq. (2a) in [1] which gives, for a non-extended dead time, the original count rate $\rho_o$ in terms of the measured quantities. An attempt to reconstruct the solution may run as follows.

With $y = {}_o\overline{R}\tau$ we can write (22) or (A4) as

$$y = -\frac{x_b}{1 + x_b} = \frac{1}{\kappa(1 + x_b)} \ln \left\{\frac{1 + x_b + x_o}{1 + x_b + \Lambda x_o}\right\} ,$$

or

$$\ln \left\{\frac{1 + x_b + x_o}{1 + x_b + \Lambda x_o}\right\} = \kappa (1 + x_b) \left[y - \frac{x_b}{1 + x_b}\right] \equiv C . \quad \text{(A17)}$$

Therefore

$$e^C = \frac{1 + x_b + x_o}{1 + x_b + \Lambda x_o} ,$$

$$x_o\ (\Lambda e^C - 1) = (1 + x_b)\ (1 - e^C) ,$$

or

$$x_o = (1 + x_b)\ \left[\frac{1 - e^C}{e^{C-\kappa} - 1}\right] . \qquad (A18)$$

With $D \equiv \kappa - C$ we finally arrive at the expression

$$x_o = (1 + x_b)\ \left[\frac{e^D - e^\kappa}{1 - e^D}\right] \qquad (A19)$$

where, since one can also write

$$C = \kappa\ [y(1 + x_b) - x_b] = \kappa\ [1 - (1 - y)\ (1 + x_b)] ,$$

we have

$$D = \kappa\ (1 - y)\ (1 + x_b) . \qquad (A20)$$

The relation (A19) obviously corresponds to (30) and is identical with the expression given in [1].

With the derivation of this result we end our attempt to put some meat around the solid, but bare bones originally offered by Axton and Ryves. It is to be feared, however, that the meal thus prepared with alien ingredients may have somewhat changed its taste. So we cannot but hope that the original cooks will excuse the possible admixture of new flavours.

Discussions with Mme M. Boutillon (BIPM) have resulted in a number of improvements which have been incorporated in the text.

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