Rapport BIPM-87/1

New light on powers of power series*

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Abstract

It is shown how a general power series, when raised to an arbitrary real power, can again be written in the form of a power series with new coefficients, the form of which is readily given by explicit expressions. By extending previous work and replacing insufficient mathematical tabulations, the new formulae make possible the effective handling of power series, even to a high order.

1. Introduction

Simple operations with (convergent) power series S of the type

$$S \equiv 1 + \sum_{j=1}^{\infty} a_j x^j , \qquad \text{with } x^2 < 1 , \tag{1}$$

such as S^2, S^{-3}, $S^{1/2}$ or $S^{-1/4}$, continue to be of much practical interest to anybody involved in the application of mathematics.

Since the information on this subject readily available for physicists, as for instance that given in [1], is clearly insufficient, a more general (yet still simple) description of the effects produced on power series by some elementary operations seems to be still very desirable.

In a previous note [2] we have treated the two special cases S^{-1} and $S^{1/2}$, for which formulae were given for the coefficients of the resulting new power developments (as well as explicit expressions up to order 8).

In the present report we intend to derive some more general formulae which are applicable to any power S^{α}, with α real. The two examples described before will turn out to be special cases and can serve as useful checks. No claim is made for the novelty of the results presented; the only purpose is to give practical help in lengthy calculations.

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* This report is dedicated to Pierre Carré on the occasion of his retirement from BIPM.

2. General approach

The new approach is based on a suggestion which P. Carré, of BIPM, made to the author as a comment on [2], probably in spring 1984. In the meantime, the need to have available explicit expressions for powers of S has turned up again in our work in various contexts (as for instance in [3]), motivating us thereby to have a new look at the general problem.

Carré's idea is to write S in the form

$$S = 1 + T , \tag{2}$$

with $$T = \sum_{j=1}^{\infty} a_j x^j .$$

At first, this may seem too simple an approach to be of much use, but this impression is wrong. If we consider an arbitrary power α of S (with α real), then we have, by virtue of the generalized binomial theorem,

$$S^\alpha = 1 + \sum_{n=1}^{\infty} \binom{\alpha}{n} T^n = 1 + \sum_{k=1}^{\infty} b_k x^k . \tag{3}$$

The first sum is finite if α is a natural number.

Our aim will now be to arrive at a general expression for the new coefficients b_k.

Let us assume for T^n the development

$$T^n = \sum_{k=n}^{\infty} {}_nc_k x^k . \tag{4}$$

If the new coefficients ${}_nc_k$ are known - their evaluation is the subject of the following section -, then we obtain from (3) and (4)

$$S^\alpha = 1 + \sum_{n=1}^{\infty} \binom{\alpha}{n} \sum_{k=n}^{\infty} {}_nc_k x^k = 1 + \sum_{k=1}^{\infty} \left[\sum_{n=1}^{k} \binom{\alpha}{n} {}_nc_k \right] x^k ,$$

since ${}_nc_k = 0$ for $n > k$. Therefore, we arrive at a general relation for the coefficients b_k appearing in the series expansion (3) of S^α, namely at

$$b_k = \sum_{n=1}^{k} \binom{\alpha}{n} {}_nc_k . \tag{5}$$

This simple formula is the main result of the present study. It allows us to split up the general problem of determining S^α into two separate parts, namely the practical evaluation of the generalized binomial coefficients $\binom{\alpha}{n}$ and the determination of the powers T^n, where n is a positive integer. It will be seen from what follows that these two partial problems are easily solved. That their results can be so readily combined to yield the coefficients b_k for the series development of S^α, as expressed by (5), is only made possible by Carré's decomposition (2) which is thus recognized as the long sought-for missing link.

3. Evaluation of T^n

Let us begin with the second problem, i.e. the transition from

$$T = \sum_{j=1}^{\infty} a_j x^j \quad \text{to} \quad T^n = \sum_{k=n}^{\infty} {}_nc_k \, x^k .$$

It is known from the theory of multinomial expansions how the new coefficients ${}_nc_k$ can be determined in terms of a_j, namely by applying the prescription

$$ {}_nc_k = \sum_{(n)} n! \prod_{j=1}^{k} \frac{a_j^{n_j}}{\prod_j n_j!} , \tag{6a}$$

where the sum (n) has to be extended over all arrangements $a_1^{n_1} a_2^{n_2} \ldots a_k^{n_k}$, with integer powers $0 \leqslant n_j \leqslant n$, which fulfil the two conditions

$$\sum_{j=1}^{k} n_j = n \quad \text{and} \quad \sum_{j=1}^{k} j\, n_j = k. \tag{6b}$$

Thus, for instance, for n = 3 and k = 6, the only arrangements $\prod_j a_j^{n_j}$ compatible with (6b) are

$$a_1^2 \, a_4, \quad a_1 \, a_2 \, a_3 \quad \text{and} \quad a_2^3 .$$

It is practical to have the coefficients ${}_nc_k$ readily available since, as a result of (5), they turn up in all our problems. This is why their explicit form has been determined and they are assembled in Table 1 (up to k = 10).

In view of the central role they play, it is important to dispose of checks allowing us to attest the reliability of the listed coefficients ${}_nc_k$. This can be easily accomplished in the following way. First, every term must obviously fulfil the two conditions imposed by (6b). As for the number of terms $p_n(k)$ from which ${}_nc_k$ is composed, we note that it corresponds to the number of partitions of k into n parts. In addition, we may point out that

$$p(k) = \sum_{n=1}^{k} p_n(k) \tag{7}$$

is the number of unrestricted partitions of k, i.e. the number of decompositions of k into integers (without regard to order). All these quantities are listed (up to k = 12) in Table 2. Finally, as for the numerical coefficients appearing in a given expression for ${}_nc_k$, we note that their sum can be shown to be equal to $\binom{k-1}{n-1}$, which is the number of ways that k objects can be placed in n boxes, when none of them is left empty. This result has already been indicated in [4].

Table 1 - Explicit form of the coefficients ${}_nc_k$ which appear in (4), for $n \leqslant k \leqslant 10$, in terms of a_j

n	k	${}_nc_k$
1	k	a_k
2	2	a_1^2
	3	$2a_1a_2$
	4	$2a_1a_3 + a_2^2$
	5	$2a_1a_4 + 2a_2a_3$
	6	$2a_1a_5 + 2a_2a_4 + a_3^2$
	7	$2a_1a_6 + 2a_2a_5 + 2a_3a_4$
	8	$2a_1a_7 + 2a_2a_6 + 2a_3a_5 + a_4^2$
	9	$2a_1a_8 + 2a_2a_7 + 2a_3a_6 + 2a_4a_5$
	10	$2a_1a_9 + 2a_2a_8 + 2a_3a_7 + 2a_4a_6 + a_5^2$
3	3	a_1^3
	4	$3a_1^2a_2$
	5	$3a_1^2a_3 + 3a_1a_2^2$
	6	$3a_1^2a_4 + 6a_1a_2a_3 + a_2^3$
	7	$3a_1^2a_5 + 6a_1a_2a_4 + 3a_1a_3^2 + 3a_2^2a_3$
	8	$3a_1^2a_6 + 6a_1a_2a_5 + 6a_1a_3a_4 + 3a_2^2a_4 + 3a_2a_3^2$
	9	$3a_1^2a_7 + 6a_1a_2a_6 + 6a_1a_3a_5 + 3a_1a_4^2 + 3a_2^2a_5 + 6a_2a_3a_4 + a_3^3$
	10	$3a_1^2a_8 + 6a_1a_2a_7 + 6a_1a_3a_6 + 6a_1a_4a_5 + 3a_2^2a_6 + 6a_2a_3a_5 + 3a_2a_4^2 + 3a_3^2a_4$

Table 1 (cont'd)

n	k	${}_nc_k$
4	4	a_1^4
	5	$4a_1^3a_2$
	6	$4a_1^3a_3 + 6a_1^2a_2^2$
	7	$4a_1^3a_4 + 12a_1^2a_2a_3 + 4a_1a_2^3$
	8	$4a_1^3a_5 + 12a_1^2a_2a_4 + 6a_1^2a_3^2 + 12a_1a_2^2a_3 + a_2^4$
	9	$4a_1^3a_6 + 12a_1^2a_2a_5 + 12a_1^2a_3a_4 + 12a_1a_2^2a_4 + 12a_1a_2a_3^2 + 4a_2^3a_3$
	10	$4a_1^3a_7 + 12a_1^2a_2a_6 + 12a_1^2a_3a_5 + 6a_1^2a_4^2 + 12a_1a_2^2a_5$
		$+ 24a_1a_2a_3a_4 + 4a_1a_3^3 + 4a_2^3a_4 + 6a_2^2a_3^2$
5	5	a_1^5
	6	$5a_1^4a_2$
	7	$5a_1^4a_3 + 10a_1^3a_2^2$
	8	$5a_1^4a_4 + 20a_1^3a_2a_3 + 10a_1^2a_2^3$
	9	$5a_1^4a_5 + 20a_1^3a_2a_4 + 10a_1^3a_3^2 + 30a_1^2a_2^2a_3 + 5a_1a_2^4$
	10	$5a_1^4a_6 + 20a_1^3a_2a_5 + 20a_1^3a_3a_4 + 30a_1^2a_2^2a_4 + 30a_1^2a_2a_3^2$
		$+ 20a_1a_2^3a_3 + a_2^5$
6	6	a_1^6
	7	$6a_1^5a_2$
	8	$6a_1^5a_3 + 15a_1^4a_2^2$
	9	$6a_1^5a_4 + 30a_1^4a_2a_3 + 20a_1^3a_2^3$
	10	$6a_1^5a_5 + 30a_1^4a_2a_4 + 15a_1^4a_3^2 + 60a_1^3a_2^2a_3 + 15a_1^2a_2^4$
7	7	a_1^7
	8	$7a_1^6a_2$
	9	$7a_1^6a_3 + 21a_1^5a_2^2$
	10	$7a_1^6a_4 + 42a_1^5a_2a_3 + 35a_1^4a_2^3$

Table 1 (cont'd)

n	k	${}_nc_k$
8	8	a_1^8
	9	$8a_1^7a_2$
	10	$8a_1^7a_3 + 28a_1^6a_2^2$
9	9	a_1^9
	10	$9a_1^8a_2$
10	10	a_1^{10}

4. General formulae for ${}_nc_k$

A closer look at Table 1 reveals a number of similarities between coefficients ${}_nc_k$ for which the difference k-n is the same. This suggests an arrangement in terms of ${}_nc_{n+r}$, with r = 0, 1, 2, ... If this is done, it is easy to see that the observed regularities can be described by the following relations

$$\begin{aligned}
{}_nc_n &= a_1^n , \\
{}_nc_{n+1} &= n\, a_1^{n-1}\, a_2 , \\
{}_nc_{n+2} &= n\, a_1^{n-1}\, a_3 + \binom{n}{2} a_1^{n-2}\, a_2^2 , \\
{}_nc_{n+3} &= n\, a_1^{n-1}\, a_4 + 2\binom{n}{2} a_1^{n-2}\, a_2a_3 + \binom{n}{3} a_1^{n-3}\, a_2^3 , \\
{}_nc_{n+4} &= n\, a_1^{n-1}\, a_5 + \binom{n}{2} a_1^{n-2}\, (2a_2a_4 + a_3^2) \\
&\quad + 3\binom{n}{3} a_1^{n-3}\, a_2^2a_3 + \binom{n}{4} a_1^{n-4}\, a_2^4 , \\
&\text{etc.}
\end{aligned} \tag{8}$$

For a further compactification of these expressions, see the Appendix.

Table 2 - The number $p_n(k)$ of partitions of k into n parts and of unrestricted partitions p(k), for $1 \leqslant k \leqslant 12$. An inspection of this table leads to the conjecture that $p_n(k) = p(k-n)$ for $n \geqslant k/2$, assuming $p(0) = 1$.

k	$p_n(k)$												p(k)
	n = 1	2	3	4	5	6	7	8	9	10	11	12	
1	1												1
2	1	1											2
3	1	1	1										3
4	1	2	1	1									5
5	1	2	2	1	1								7
6	1	3	3	2	1	1							11
7	1	3	4	3	2	1	1						15
8	1	4	5	5	3	2	1	1					22
9	1	4	7	6	5	3	2	1	1				30
10	1	5	8	9	7	5	3	2	1	1			42
11	1	5	10	11	10	7	5	3	2	1	1		56
12	1	6	12	15	13	11	7	5	3	2	1	1	77

5. Integer powers of S

a) Positive exponents

A simple but particularly important special case of S^α is given when the exponent α is a natural number m = 1, 2, 3, ... We then have from (5)

$$b_k = \sum_{n=1}^{k} \binom{m}{n} {}_n c_k , \qquad (9)$$

with ${}_n c_k$ as given in Table 1. Since the numerical values of the binomial coefficients are readily available, we just use their traditional notation. The explicit form of b_k, in terms of a_j, can therefore be obtained as

$$
\begin{aligned}
b_1 &= \binom{m}{1} a_1 , \\
b_2 &= \binom{m}{1} a_2 + \binom{m}{2} a_1^2 , \\
b_3 &= \binom{m}{1} a_3 + 2\binom{m}{2} a_1 a_2 + \binom{m}{3} a_1^3 , \\
b_4 &= \binom{m}{1} a_4 + \binom{m}{2} (2a_1a_3 + a_2^2) + 3\binom{m}{3} a_1^2 a_2 + \binom{m}{4} a_1^4 , \\
b_5 &= \binom{m}{1} a_5 + 2\binom{m}{2} (a_1a_4 + a_2a_3) + 3\binom{m}{3} (a_1^2a_3 + a_1a_2^2) \\
&\qquad + 4\binom{m}{4} a_1^3 a_2 + \binom{m}{5} a_1^5 , \\
b_6 &= \binom{m}{1} a_6 + \binom{m}{2} (2a_1a_5 + 2a_2a_4 + a_3^2) + \binom{m}{3} (3a_1^2a_4 + 6a_1a_2a_3 + a_2^3) \\
&\qquad + \binom{m}{4} (4a_1^3a_3 + 6a_1^2a_2^2) + 5\binom{m}{5} a_1^4 a_2 + \binom{m}{6} a_1^6 ,
\end{aligned}
\tag{10}
$$

etc.

Since $\binom{m}{k} = 0$ for $k > m$, some of the terms listed above actually vanish.

b) Negative exponents

Whereas, for m = 1, 2, 3, ..., there is the well-known relation

$$\binom{m}{n} = \frac{m\,(m-1)\,(m-2)\,\ldots\,(m-n+1)}{n!} = \frac{m!}{(m-n)!\;n!} , \tag{11}$$

we now have to see what happens to the analogous binomial coefficient $\binom{-m}{n}$ where both m and n are natural numbers, remembering that factorials of negative arguments diverge. However, we can still write

$$
\begin{aligned}
\binom{-m}{n} &= \frac{(-m)(-m-1)(-m-2)\,\ldots\,(-m-n+1)}{n!} \\
&= \frac{(-1)^n}{n!}\,\frac{(m+n-1)!}{(m-1)!} = (-1)^n \binom{m+n-1}{n} .
\end{aligned}
\tag{12}
$$

The two expressions (11) and (12) look quite different. Is there a way to combine them into a common formula? This can indeed be achieved by introducing a "cut-off operator" of the form

$$(x)_+ \equiv \begin{cases} x , & \text{for} \quad x \geqslant 0 , \\ 0 , & \text{"} \quad x \leqslant 0 . \end{cases} \tag{13}$$

It is not difficult to verify that with this notation the expression

$$\binom{\pm m}{n} = \frac{(\pm 1)^n}{n!}\,\frac{\left[m + (\mp n \pm 1)_+\right]!}{\left[m - n + (\mp n \pm 1)_+\right]!} \tag{14}$$

is indeed a valid generalization of (11) and (12). Yet, it must be admitted that this general formula looks rather artificial and we therefore would not expect it to be of frequent practical use - although it could be very easily programmed on a computer.

If the binomial coefficients $\binom{m}{n}$ in (10) are replaced by $\binom{-m}{n}$, then all the explicit relations (10) are still valid, provided they are interpreted according to (12) or (14).

Important special cases concern m = 1, 2 or 3, for which we readily obtain from (12)

$$\binom{-1}{n} = \frac{(-1)^n}{n!}\,\frac{n!}{0!} = (-1)^n ,$$

$$\binom{-2}{n} = \frac{(-1)^n}{n!}\,\frac{(n+1)!}{1!} = (-1)^n\,(n+1) , \tag{15}$$

$$\binom{-3}{n} = \frac{(-1)^n}{n!}\,\frac{(n+2)!}{2!} = (-1)^n\,\frac{(n+1)(n+2)}{2} .$$

We note that the first of these relations has already found (unconsciously) an application in [2], namely for the evaluation of S^{-1}, although at that time this was not seen in the present light.

6. Reciprocal integers as exponents

Let us consider another special case, namely when the exponent is of the form $\alpha = \pm\,1/m$, with m a natural number $\geqslant 2$. We look, in particular, for an explicit expression of $\binom{\alpha}{n}$ which appears in (5) and is thus needed for the evaluation of the new coefficients b_k. According to its definition, the binomial coefficient, for $\alpha = +\,1/m$, is given by

$$\binom{1/m}{n} = \frac{\left(\frac{1}{m}\right)\left(\frac{1-m}{m}\right)\left(\frac{1-2m}{m}\right)\,\cdots\,\left(\frac{1-m(n-1)}{m}\right)}{n!}$$

$$= (-1)^{n-1}\,\frac{1(m-1)(2m-1)\,\cdots\,(nm-m-1)}{m^n\;n!}$$

$$= \frac{(-1)^{n-1}}{m^n\;n!}\,(nm-m-1)!\tilde{m}! , \tag{16a}$$

where $(x)!\tilde{m}! \equiv x(x-m)(x-2m)\,\cdots\,x'$, with $0 < x' \leqslant m$, is the m-fold factorial of $x > 0$.

Likewise, we have for $\alpha = -\,1/m$

$$\binom{-1/m}{n} = \frac{\left(\frac{1}{-m}\right)\left(\frac{m+1}{-m}\right)\left(\frac{2m+1}{-m}\right)\,\cdots\,\left(\frac{nm-m+1}{-m}\right)}{n!}$$

$$= \frac{(-1)^n}{m^n\;n!}\,(nm-m+1)!\tilde{m}! . \tag{16b}$$

Therefore, the binomial coefficient for $\alpha = \pm\,1/m$ can be written as

$$\binom{\pm 1/m}{n} = \mp\,(-1)^n\,\frac{(nm-m\mp 1)!\tilde{m}!}{(nm)!\tilde{m}!} , \tag{17}$$

with the definition $(-1)!\tilde{m}! = 1$, for $m \geqslant 2$.

A simple illustration may be in order.

For $\alpha = 1/2$ application of (16a) or (17) yields

$$\binom{1/2}{n} = (-1)^{n-1} \frac{(2n-3)!!}{(2n)!!} ,$$

which explains the somewhat mysterious expression that we have found in [2] for the coefficients b_k in the case of a square root of S. Apparently, it was the observation that $\binom{1/2}{n}$ appears in [2] for b_k which has led P. Carré to his decomposition (2).

Other simple explicit results are, for example,

$$\binom{-1/2}{n} = (-1)^{n} \frac{(2n-1)!!}{(2n)!!} ,$$

$$\binom{1/3}{n} = (-1)^{n-1} \frac{(3n-4)!\tilde{3}!}{(3n)!\tilde{3}!} ,$$

$$\binom{-1/3}{n} = (-1)^{n} \frac{(3n-2)!\tilde{3}!}{(3n)!\tilde{3}!} ,$$

$$\binom{1/4}{n} = (-1)^{n-1} \frac{(4n-5)!\tilde{4}!}{(4n)!\tilde{4}!} , \qquad (18)$$

$$\binom{-1/4}{n} = (-1)^{n} \frac{(4n-3)!\tilde{4}!}{(4n)!\tilde{4}!} ,$$

$$\binom{1/5}{n} = (-1)^{n-1} \frac{(5n-6)!\tilde{5}!}{(5n)!\tilde{5}!} ,$$

$$\binom{-1/5}{n} = (-1)^{n} \frac{(5n-4)!\tilde{5}!}{(5n)!\tilde{5}!} ,$$

For the numerical evaluation it is practical to make use of the corresponding recursion formulae which are

$$\binom{\pm 1/m}{1} = \pm \frac{1}{m} \quad \text{and}$$

$$\binom{\pm 1/m}{n} = - \left[\frac{(n-1)m \mp 1}{nm}\right] \binom{\pm 1/m}{n-1} , \qquad \text{for } n \geq 2 . \qquad (19)$$

Some numerical values are listed in Table 3.

By virtue of (5) all these expressions, together with the coefficient ${}_nc_k$ from Table 1, now yield immediately the coefficients b_k appearing in the corresponding series.

Table 3 - Numerical values for some binomial coefficients of the form $\binom{\pm 1/m}{n}$

n	$\binom{1/2}{n}$	$\binom{-1/2}{n}$	$\binom{1/3}{n}$	$\binom{-1/3}{n}$	$\binom{1/4}{n}$	$\binom{-1/4}{n}$	$\binom{1/5}{n}$	$\binom{-1/5}{n}$
1	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{1}{5}$	$\frac{-1}{5}$
2	$\frac{-1}{8}$	$\frac{3}{8}$	$\frac{-1}{9}$	$\frac{2}{9}$	$\frac{-3}{32}$	$\frac{5}{32}$	$\frac{-2}{25}$	$\frac{3}{25}$
3	$\frac{1}{16}$	$\frac{-5}{16}$	$\frac{5}{81}$	$\frac{-14}{81}$	$\frac{7}{128}$	$\frac{-15}{128}$	$\frac{6}{125}$	$\frac{-11}{125}$
4	$\frac{-5}{128}$	$\frac{35}{128}$	$\frac{-10}{243}$	$\frac{35}{243}$	$\frac{-77}{2\ 048}$	$\frac{195}{2\ 048}$	$\frac{-21}{625}$	$\frac{44}{625}$
5	$\frac{7}{256}$	$\frac{-63}{256}$	$\frac{22}{729}$	$\frac{-91}{729}$	$\frac{231}{8\ 192}$	$\frac{-663}{8\ 192}$	$\frac{399}{15\ 625}$	$\frac{-924}{15\ 625}$
6	$\frac{-21}{1\ 024}$	$\frac{231}{1\ 024}$	$\frac{-154}{6\ 561}$	$\frac{728}{6\ 561}$	$\frac{-1\ 463}{65\ 536}$	$\frac{4\ 641}{65\ 536}$	$\frac{-1\ 596}{78\ 125}$	$\frac{4\ 004}{78\ 125}$
7	$\frac{33}{2\ 048}$	$\frac{-429}{2\ 048}$	$\frac{374}{19\ 683}$	$\frac{-1\ 976}{19\ 683}$	$\frac{4\ 807}{262\ 144}$	$\frac{-16\ 575}{262\ 144}$	$\frac{6\ 612}{390\ 625}$	$\frac{-17\ 732}{390\ 625}$
8	$\frac{-429}{32\ 768}$	$\frac{6\ 435}{32\ 768}$	$\frac{-935}{59\ 049}$	$\frac{5\ 434}{59\ 049}$	$\frac{-129\ 789}{8\ 388\ 608}$	$\frac{480\ 675}{8\ 388\ 608}$	$\frac{-28\ 101}{1\ 953\ 125}$	$\frac{79\ 794}{1\ 953\ 125}$

As a simple application of the procedure, let us determine the explicit form of the coefficients b_k for $\alpha = -1/2$, hence of

$$b_k = \sum_{n=1}^{k} \binom{-1/2}{n} \, {}_n c_k \, .$$

They are readily found to be (for $k \leqslant 8$)

$$b_1 = -\frac{1}{2} a_1 ,$$

$$b_2 = -\frac{1}{2} a_2 + \frac{3}{8} a_1^2 ,$$

$$b_3 = -\frac{1}{2} a_3 + \frac{3}{4} a_1 a_2 - \frac{5}{16} a_1^3 ,$$

$$b_4 = -\frac{1}{2} a_4 + \frac{3}{8} (2a_1a_3 + a_2^2) - \frac{15}{16} a_1^2 a_2 + \frac{35}{128} a_1^4 ,$$

$$b_5 = -\frac{1}{2} a_5 + \frac{3}{4} (a_1a_4 + a_2a_3) - \frac{15}{16} (a_1^2a_3 + a_1a_2^2) + \frac{35}{32} a_1^3 a_2 - \frac{63}{256} a_1^5 ,$$

$$b_6 = -\frac{1}{2} a_6 + \frac{3}{8} (2a_1a_5 + 2a_2a_4 + a_3^2) - \frac{5}{16} (3a_1^2a_4 + 6a_1a_2a_3 + a_2^3) + \frac{35}{64} (2a_1^3a_3 + 3a_1^2a_2^2) - \frac{315}{256} a_1^4 a_2 + \frac{231}{1\,024} a_1^6 , \tag{20}$$

$$b_7 = -\frac{1}{2} a_7 + \frac{3}{4} (a_1a_6 + a_2a_5 + a_3a_4) - \frac{15}{16} (a_1^2a_5 + 2a_1a_2a_4 + a_1a_3^2 + a_2^2a_3) + \frac{35}{32} (a_1^3a_4 + 3a_1^2a_2a_3 + a_1a_2^3) - \frac{315}{256} (a_1^4a_3 + 2a_1^3a_2^2) + \frac{693}{512} a_1^5 a_2 - \frac{429}{2\,048} a_1^7 ,$$

$$b_8 = -\frac{1}{2}a_8 + \frac{3}{8}(2a_1a_7 + 2a_2a_6 + 2a_3a_5 + a_4^2)$$
$$- \frac{15}{16}(a_1^2a_6 + 2a_1a_2a_5 + 2a_1a_3a_4 + a_2^2a_4 + a_2a_3^2)$$
$$+ \frac{35}{128}(4a_1^3a_5 + 12a_1^2a_2a_4 + 6a_1^2a_3^2 + 12a_1a_2^2a_3 + a_2^4)$$
$$- \frac{315}{256}(a_1^4a_4 + 4a_1^3a_2a_3 + 2a_1^2a_2^3)$$
$$+ \frac{693}{1\,024}(2a_1^5a_3 + 5a_1^4a_2^2) - \frac{3\,003}{2\,048}a_1^6a_2 + \frac{6\,435}{32\,768}a_1^8 .$$

7. Other exponents

Exponents α which are larger than unity and not integers give rise to generalized binomial coefficients which are somewhat more cumbersome to handle. We can easily see the problem by treating a special case, for instance $\alpha = 5/2$. Here we have

$$\binom{5/2}{n} = \frac{\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\,(\frac{-1}{2})\,(\frac{-3}{2})\,\ldots\,(\frac{5}{2}-n+1)}{n!} .$$

For $n \geq 4$ we have, since $\frac{5}{2} - n+1 = \frac{-(2n-7)}{2}$,

$$\binom{5/2}{n} = (-1)^{n-3}\,\frac{5!!}{2^n\,n!}\,(2n-7)!! ,$$

but this formula clearly does not hold for $n \leq 3$.

If we apply the notation introduced in (13), then an expression, valid for any value of $n \geq 1$, can be given in the form

$$\binom{5/2}{n} = (-1)^{(n-3)_+}\,\frac{5!!}{(2n)!!}\,\frac{(2n-7)_+!!}{(5-2n)_+!!} , \qquad (21)$$

but this again is rather complicated.

It is readily seen that negative values of α are much easier to treat. Thus, for instance for $\alpha = -5/2$ we find

$$\binom{-5/2}{n} = \frac{(\frac{-5}{2})\,(\frac{-7}{2})\,(\frac{-9}{2})\,\ldots\,(\frac{3+2n}{-2})}{n!}$$
$$= (-1)^n\,\frac{(2n+3)!!}{3(2n)!!} , \quad \text{for any } n \geq 1 . \qquad (22)$$

It would certainly be possible to indicate a general expression valid for any value of α, but in the absence of a real need we prefer to leave the pleasure of deriving such a formula to the reader.

It will be obvious that for all practical applications the complications mentioned above can be easily avoided by two successive applications of some of the simpler formulae given before since we still have, of course, identities like

$$S^{\pm m/n} = [S^{\pm m}]^{1/n} = [S^{\pm 1/n}]^m ,$$

where m and n are positive integers.

Once more Madame M. Boutillon is to be thanked for her careful and critical reading of an early draft; the present version has taken into account most of her suggestions.

The dedication of this report to P. Carré would be easily justified by his very substantial contribution, but in reality it has a deeper meaning. For more than a quarter of a century he has been a prominent figure at BIPM. His great intellectual powers were not always easy to accept for everybody. Perhaps he reached his best achievements, in criticizing and improving the work of others. His strong demand for clarity and truth, coupled with a pedagogical disposition, made him an implacable opponent of any form of mental indolence. As such a turn of mind has become rare these days, Carré will soon be missed at BIPM.

Appendix

After finishing the present report, we happened to come across a general expression in [5] which not only seems to be in line with the set of equations given in (8), but also allows us to develop them for higher values of r and at the same time to get some clearer insight into their combinatorial structure. This relation, when written in our present notation, has the simple form (for $r \geq 1$)

$$_n c_{n+r} = Y_r / r! , \qquad \text{(A1)}$$

where Y_r are the so-called Bell polynomials $Y_r(f_1,g_1, f_2,g_2, \ldots, f_r,g_r)$, with

$$f_j = (n)_j \, a_1^{n-j} \quad \text{and} \quad g_j = j! \, a_{j+1} , \qquad \text{(A2)}$$

and where the "falling factorials" are defined by

$$(n)_j \equiv \frac{n!}{(n-j)!} = n\,(n-1)\,(n-2)\,\ldots\,(n-j+1) , \quad \text{for } n \geq j. \qquad \text{(A3)}$$

The Bell polynomials are listed for $1 \leq r \leq 8$ in [5] and, with another notation, up to r = 10 in [6]. The case r = 0 could be included by putting $Y_o = f_o$.

As the result (A1) is only given incidentally in the context of a problem and without proof, a verification may be welcome. Let us choose r = 4. Since, according to [5] or [6],

$$Y_4 = f_1 g_4 + f_2(4g_1 g_3 + 3g_2^2) + f_3(6g_1^2 g_2) + f_4 g_1^4 , \qquad (A4)$$

substitution of (A2) leads to

$$\begin{aligned} {}_nc_{n+4} &= Y_4/4! \\ &= \frac{1}{24}\left[24na_1^{n-1}\, a_5 + n(n-1)a_1^{n-2}\, (24a_2a_4 + 12a_3^2) \right. \\ &\qquad \left. + n(n-1)(n-2)a_1^{n-3}\, (12a_2^2a_3) + n(n-1)(n-2)(n-3)a_1^{n-4}\, a_2^4\right] \\ &= na_1^{n-1}a_5 + \binom{n}{2}a_1^{n-2}(2a_2a_4 + a_3^2) + \binom{n}{3}3a_1^{n-3}a_2^2a_3 + \binom{n}{4}a_1^{n-4}a_2^4, \end{aligned} \qquad (A5)$$

in agreement with the corresponding relation in (8). While the compactness of (A1) is quite impressive, it will clearly be more practical to use the explicit form given in Table 1 whenever the coefficients ${}_nc_k$ required for numerical calculations are listed there.

The existence of the general formula (A1) is in line with our feeling that probably none of the relations given here is actually new, but we still think that some of the explicit expressions stated in this report may be of some use in practical calculations.

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(February 1987)