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How well can we realize a generalized dead time? *

Part I: For an unperturbed Poisson process

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Abstract

In our practical realization of a generalized dead time, the type applicable to an arriving pulse depends on a periodic signal of frequency ν with two states. It is shown that the independence of subsequent choices, for Poisson arrivals of events, can be guaranteed as long as ν exceeds the average count rate of the process under study.

1. Introduction

The notion of the traditional two types of dead times, called non-extended and extended, has been generalized in 1953 by Albert and Nelson [1]. In their model, one or the other of the usual types is chosen at random, but with a certain probability, for each incoming pulse. If we denote by θ the probability that an extended (E) dead time is chosen, i.e.

$$P_\theta(E) = \theta , \quad \text{with} \quad 0 \leqslant \theta \leqslant 1 , \tag{1a}$$

then we have obviously for the non-extended (N) type

$$P_\theta(N) = 1 - \theta . \tag{1b}$$

An important feature of this model is that subsequent choices are assumed to be completely independent of each other.

Originally, the motivation for this generalization clearly came from theoretical considerations: it made possible a unified treatment of dead-time losses, and the traditional two types had now become limiting cases of a more general selection rule describing the "survival" of pulses, namely for $\theta = 0$ and $\theta = 1$.

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* This report is dedicated to Peter J. Campion on the occasion of his retirement from the National Physical Laboratory, Teddington.

A subsequent inquiry into the properties of a generalized dead time has brought to light several features which made it desirable to have such a device available for actual experimentation - even if this should only be possible in some approximate form. Is there any real hope for a practical implementation of the model?

Among the various questions arising in an attempt to realize a system with properties as close as possible to the mathematical model of Albert and Nelson, we may mention three that are of obvious relevance, namely

a) How can we perform a random choice of the types?
b) How can θ be accurately measured, and in which time?
c) How can we guarantee the independence of subsequent choices?

As for a) the answer is that we do not try to make a truly random choice, not even by the use of pseudo-random numbers, as this would require an on-line computer. In fact, our choices are of a very poor nature since they are based on a strictly periodic signal. This primitive solution calls for some justification, which is in essence the subject of the present report. Whereas speed and simplicity of such an approach are obvious from the start, the surprisingly close approximation to a completely random behaviour - anticipating the outcome of this study - is due to the statistical nature of the pulse arrivals. This will also give an answer to c), because the independence of the choices of the dead-time type for subsequent pulses can be evaluated exactly (for a given set of parameters). More details concerning the electronic realization made at the BIPM as well as on b) will be given in another study [2].

In the following we derive an expression which will permit one to judge in a quantitative way the degree of approximation to the ideal situation (1) that can be achieved in the present simple approach. It is easy to see in advance that the assumed independence of consecutive choices will only be poorly realized when the count rates become comparable to, or even exceed, the frequency of the binary signal.

2. Evaluation of a conditional probability

It is clear, first, that the study can be limited to the case of two consecutive pulses to which the whole process can be readily reduced. In addition, it is easy to see that the analysis of one out of the four possible sequences of types that can be formed with two pulses is sufficient, since the others are then given by elementary symmetry relations.

The basic situation is sketched in Fig. 1. A periodic signal h with two states (which we may call "0" and "1"), of frequency $\nu = 1/T$, determines, by means of appropriate electronic gates, which of the two types of dead time will actually be chosen at the moment of a pulse arrival. Let "0" correspond to N and "1" to E, for example. We start with a random arrival of a pulse (at t_0) and ask for the next event to fall in a period which corresponds to the opposite type, for instance.

Since the density for the time interval between subsequent events in a Poisson process of count rate ρ is known to be of exponential form, we have, by reasoning in time units of T, the expression

$$f(z) = r\, e^{-r(z-z_o)}, \quad \text{for } z > z_o, \tag{2}$$

where now $z = t/T$ and $r = \rho T$.

The quantity $z_o = t_o/T$ corresponds to the arrival of the first pulse.

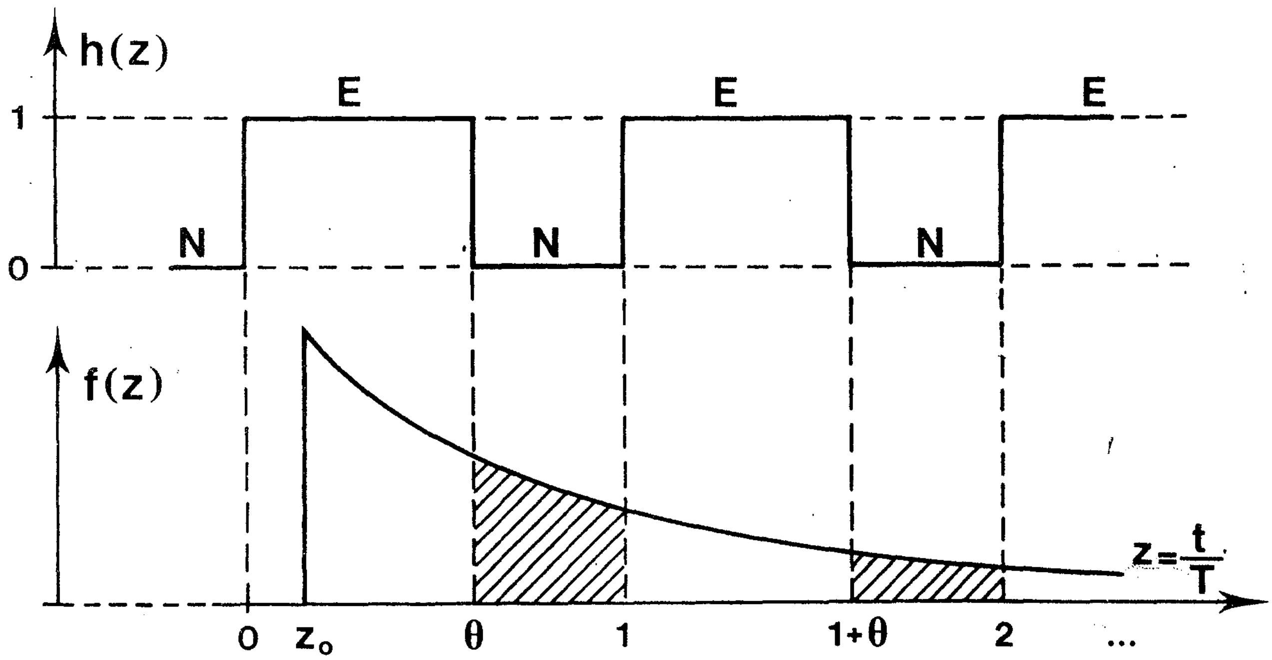

Fig. 1: Schematic representation of the relation between the periodic signal h(z), which determines the type of dead time chosen (E or N) at a given moment, and the density f(z) for the arrival of the next random pulse (see text).

Let us start with a pulse falling in the region E (at z_o) and determine the conditional probability that the next arrival, for which the density is given by f(z), will fall in a time region N. This is readily seen to be given by (hatched region in Fig. 1)

$$P_\theta(N|E,z_o) = \int_\theta^1 f(z)\,dz + \int_{1+\theta}^2 f(z)\,dz + \int_{2+\theta}^3 f(z)\,dz + \ldots . \tag{3}$$

Since

$$\int_{j+\theta}^{j+1} f(z)\,dz = r \int_{j+\theta}^{j+1} e^{-r(z-z_o)}\,dz = e^{-r(j+\theta-z_o)} - e^{-r(j+1-z_o)},$$

we have

$$P_\theta(N|E,z_o) = e^{rz_o} \sum_{j=0}^{\infty} [e^{-r(j+\theta)} - e^{-r(j+1)}]$$

$$= e^{rz_o} [\frac{e^{-r\theta} - e^{-r}}{1 - e^{-r}}] . \qquad (4)$$

As the arrival time of the first event is actually unknown, we have to average over all possibilities in order to find the unconditional probability for a sequence of types E,N. This leads to

$$P_\theta(N|E) = \frac{1}{\theta} \int_o^\theta P_\theta(N|E, z_o)\, dz_o$$

$$= \frac{1}{\theta} (\frac{e^{-r\theta} - e^{-r}}{1 - e^{-r}}) \int_o^\theta e^{rz_o}\, dz_o$$

$$= \frac{1}{r\theta} (\frac{e^{-r\theta} - e^{-r}}{1 - e^{-r}}) (e^{r\theta} - 1) . \qquad (5)$$

Let us now establish a useful symmetry. If in (5) we replace θ by the new variable $\lambda = \theta - 1/2$, we can easily find the following form

$$\frac{P_\theta(N|E)}{1 - \theta} = \frac{1}{r(\frac{1}{2} + \lambda)(\frac{1}{2} - \lambda)} [\frac{1 + e^{-r} - e^{-r/2}(e^{r\lambda} + e^{-r\lambda})}{1 - e^{-r}}]$$

$$= \frac{1}{r(\frac{1}{4} - \lambda^2)} [\frac{1 + e^{-r} - 2\, e^{-r/2} \cosh(r\lambda)}{1 - e^{-r}}] . \qquad (6)$$

Since this expression is symmetrical in λ, it follows that changing θ to $1 - \theta$ leaves (6) invariant.

3. Discussion and conclusion

Since we consider two types of dead times, there are only two possible choices for the second pulse and we thus have obviously the relation

$$P_\theta(E|E) = 1 - P_\theta(N|E) . \qquad (7a)$$

In addition, as a consequence of the periodicity of the signal h(z) and its association with the types, it is easy to see that

$$P_\theta(E|N) = P_{1-\theta}(N|E) , \tag{7b}$$

and finally also

$$P_\theta(N|N) = 1 - P_\theta(E|N) = 1 - P_{1-\theta}(N|E) = P_{1-\theta}(E|E) . \tag{7c}$$

This covers the four possible cases. As they can now all be readily compared with the sequence E,N, described explicitly by (5), it will be sufficient to proceed with this specific situation in what follows.

Obviously, one of the most important problems is to know to which degree the independence of two subsequent choices of the type of dead time can be realized. By a development of (5) up to terms r^3 we find, after some simple rearrangements,

$$P_\theta(N|E)$$

$$\cong \frac{1}{r\theta}\left(1+r\theta+\frac{r^2\theta^2}{2}+\frac{r^3\theta^3}{6}-1\right)\left[\frac{1-r\theta+\frac{r^2\theta^2}{2}-\frac{r^3\theta^3}{6}-\left(1-r+\frac{r^2}{2}-\frac{r^3}{6}\right)}{1-\left(1-r\frac{r^2}{2}-\frac{r^3}{6}\right)}\right] ,$$

$$= (1-\theta)\,[1-\Delta(\theta)] ,$$

$$\text{with}\quad \Delta(\theta) \cong \frac{1}{12} r^2\,\theta(1-\theta) . \tag{8}$$

This explicit form confirms that $\Delta(\theta)$ is unaffected by a change of θ to $1-\theta$, which is in line with the result (6).

By taking advantage of the symmetries described by (7), the situation for the four possible sequences of types can be summarized by the explicit relations

$$\begin{aligned} P_\theta(N|N) &= 1-\theta+\theta\Delta(\theta) , \\ P_\theta(N|E) &= (1-\theta)\,[1-\Delta(\theta)] , \\ P_\theta(E|N) &= \theta[1-\Delta(\theta)] , \\ P_\theta(E|E) &= \theta+(1-\theta)\,\Delta(\theta) . \end{aligned} \tag{9}$$

This can also be written in a somewhat different form by using relative differences. We then obtain

- for changing types:

$$\frac{P_\theta(N|E) - P_\theta(N)}{P_\theta(N)} = -\Delta(\theta) ,$$

$$\frac{P_\theta(E|N) - P_\theta(E)}{P_\theta(E)} = -\Delta(\theta) ; \qquad (10a)$$

- for unchanged types:

$$\frac{P_\theta(N|N) - P_\theta(N)}{P_\theta(E)} = \Delta(\theta) ,$$

$$\frac{P_\theta(E|E) - P_\theta(E)}{P_\theta(N)} = \Delta(\theta) . \qquad (10b)$$

The relative deviation of the conditional probability $P_\theta(N|E)$ from its nominal or ideal value $P_\theta(N) = 1 - \theta$ is thus given by $\Delta(\theta)$, according to (10a), and similarly for the other cases.

The largest deviation occurs for $\theta = 1/2$ where it is given by the simple approximate expression

$$\Delta_{max} = \Delta(1/2) \cong r^2/48 = \rho^2 T^2/48 , \qquad (11)$$

while the limiting values $\Delta(0)$ and $\Delta(1)$ vanish exactly.

A numerical check with the exact values deduced from the expression (5) reveals that the approximation (8) is excellent for all practical purposes (i.e. for $\rho T \lesssim 1$), and slightly pessimistic for high count rates, as can be seen from Table 1.

It follows from Table 1 (or likewise e.g. from eq. 10) that for any value of θ the simulation allows us to choose at random, with specified probability, a given dead-time type which is practically independent of the previous choice. The slight correlation, expressed qualitatively by $\Delta(\theta)$, is always less than 0.1 %, provided that we can arrange to have $\rho T \lesssim 0.2$. If a value of 0.5 % may be considered acceptable, the condition can be relaxed to $\rho T \lesssim 0.5$. For the frequency of ν = 500 kHz used at present, sources with count rates of at least up to ρ = 100 000 s^{-1} can therefore be safely handled. A higher signal frequency could be employed if needed. It should be noted that the simple simulation suggested here implies no error in the relative frequencies with which the types are chosen on the average, for, by application of the theorem of total probabilities, we still have

$$P(N) = P_\theta(N|E)\, P_\theta(E) + P_\theta(N|N)\, P_\theta(N)$$

$$= (1 - \theta)(1 - \Delta)\, \theta + (1 - \theta + \theta\Delta)(1 - \theta) = 1 - \theta ,$$

and likewise

$$P(E) = P_\theta(E|N)\, P_\theta(N) + P_\theta(E|E)\, P_\theta(E)$$

$$= \theta(1 - \Delta)(1 - \theta) + [\theta + (1 - \theta)\Delta] = \theta ,$$

as one would expect. Hence, the only (minor) drawback of the proposed simulation lies in the fact that successive choices are not completely independent of each other, as is assumed in the model. However, the remaining correlation is known (for instance by means of eqs. 9 or 10), and in particular it is always possible, as evidenced by (11), to bring this defect to a negligibly small value by choosing an appropriate value for the frequency $\nu = 1/T$ of the periodic signal which determines the type of dead time.

Table 1: Numerical comparison of the approximate formula (8) with some exact values of $\Delta(\theta)$, which indicates the relative deviation of our simulated generalized dead time from its ideal behaviour.

ρT	θ	$\Delta(\theta)$ exact	$\Delta(\theta)$ approx.
0.05	0.25	0.000 039	0.000 039
	0.50	0.000 052	0.000 052
	0.75	0.000 039	0.000 039
0.1	0.25	0.000 156	0.000 156
	0.50	0.000 208	0.000 208
	0.75	0.000 156	0.000 156
0.2	0.25	0.000 624	0.000 625
	0.50	0.000 832	0.000 833
	0.75	0.000 624	0.000 625
0.5	0.25	0.003 88	0.003 91
	0.50	0.005 18	0.005 21
	0.75	0.003 88	0.003 91
1	0.25	0.015 3	0.015 6
	0.50	0.020 3	0.020 8
	0.75	0.015 3	0.015 6
2	0.25	0.057 3	0.062 5
	0.50	0.075 8	0.083 3
	0.75	0.057 3	0.062 5

It will become clear from the results of a subsequent study [2] in which the practical measurements of the parameter θ (by a method using triple pulses) are described, that the precisions we are discussing here are quite realistic, as they can be reached in a reasonable measuring time (a few minutes). Some recent measurements of generalized dead times [3] have permitted us to confirm these expectations and they also show that the accuracy thus available is adequate for practical purposes.

The effect of a perturbation in the Poisson process on our approach to produce a generalized dead time will be studied in part II.

Guy Ratel, of BIPM, is to be thanked for a careful reading of this report.

Dr. Campion's retirement from NPL, after a distinguished career, seems to be a fitting opportunity to thank him for his long and successful association with the radioactivity work at BIPM. From 1960 to 1979 he served as a member of Section II of CCEMRI, the last ten years of this period as a very efficient chairman. His name will remain associated with the 4πβ-γ coincidence method for many generations of metrologists.

References

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[2] P. Bréonce, J.W. Müller, to be published as a Rapport BIPM

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