Equivalent dead times

(Some calculations, part I)

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Abstract

It is shown that a series arrangement of two dead times (τ_1 and τ_2), where each of them is either of the extended (E) or of the non-extended (N) type, can always be replaced by a generalized dead time (involving τ_2 and a type parameter θ) which yields the same output rate. For the sequences E-N and N-E explicit formulae are given for the evaluation of the parameter θ of the equivalent dead time.

1. Introduction

The arrangement of two dead times in series is of great practical importance, in particular for accurate activity measurements. Yet, it has received rather little attention, as is evidenced by the scarce literature dealing with this topic. This situation may be due to the fact that many experimentalists are of the opinion that the corresponding corrections can normally be neglected and are therefore not worth the trouble. While this attitude may be justified for low count rates, it becomes doubtful at higher ones.

The present knowledge concerning series arrangements of dead times is very limited. All we actually dispose of are some formulae for the output rates when the pulse sequence at the input forms a Poisson process [1]. The effect of the first (smaller) dead time can then be expressed conveniently by a transmission factor T_1, an approach which allows us to determine the original count rate in a simple way [2]. On the other hand, virtually nothing is known about the interval distribution and the counting statistics of the output process.

The recent availability of generalized dead times has given a new stimulus to the whole field of counting statistics, and the series arrangements are no exception. In particular, it is easy to see [3] that it should be possible to replace any series of two dead times (which are of the traditional types) by a single generalized dead time if we choose for its length the value of the second element. Since it then corresponds, as far as count rates are concerned, exactly to the series arrangement of two, we shall call it an equivalent dead time. The only problem that remains to be solved consists in determining the corresponding value of the parameter θ.

The advantage resulting from the replacement of a series arrangement by a (real or fictive) single dead time will be obvious: instead of having to deal with the often rather complicated expressions valid for two dead times in series, the known inversion formula for a generalized dead time or a simple iterative numerical method [4] then readily yields the desired original count rate ρ, provided that θ is known. As the determination of ρ is achieved by calculation, it will be obvious that no device capable of producing a generalized dead time has to be available for taking advantage of the present approach.

2. Method used

The approach chosen, for a given set of parameters (count rate ρ and dead times τ_1, τ_2 of specified type), to determine the generalized "type parameter" θ is straightforward. We use the formula corresponding to the series arrangement chosen (Fig. 1) which yields, for a given input ρ, the count rate R at the output. On the other hand, the output R' of a generalized dead time (with parameters τ_2 and θ) is given by the Takács formula (see for instance [5]). Therefore, equating R with R' will result in an expression for θ, the only free parameter.

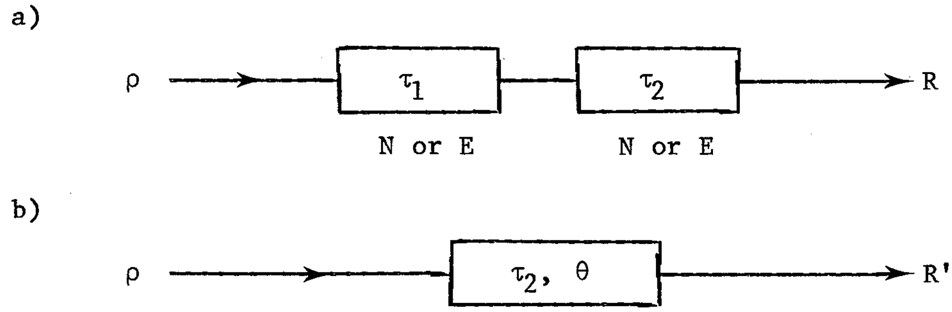

Fig. 1. Schematic arrangement of dead times and the corresponding count rates.

a) Two dead times in series (any combination of types), with $\tau_1 \leqslant \tau_2$.
b) A single generalized dead time. If θ is chosen such that R' = R, this dead time is said equivalent to the corresponding series arrangement of a).

A possible practical way of finding this "equivalent" value of θ would be to apply numerical methods. However, in order to obtain functional relations - although these will in general only be approximate ones - we prefer in what follows to use series developments.

It is well known from the equations (52) to (57) given in [1] that the expressions for R are simple only for the two "mixed" arrangements (namely N-E and E-N), but rather complicated for the "pure" cases N-N and E-E. In this first part we shall restrict ourselves to the two simple cases, leaving the analogous developments for N-N and E-E to a subsequent study. The difficult problems concerning the convergence of our series developments as well as pertinent other mathematical subtleties will not be touched upon in what follows.

For the generalized dead time we recall the relation [5]

$$R' = \frac{\theta \rho}{e^{\theta\rho\tau_2} + \theta - 1} , \tag{1}$$

which will be used in the form of a series development of $x = \rho\,\tau_2$, for which a straightforward calculation gives

$$\frac{\rho}{R'} = 1 + x + \sum_{j=2}^{\infty} \frac{x^j}{j!}\,\theta^{j-1} . \tag{2}$$

3. <u>The case E-N</u>

For the situation where the first dead time τ_1 is of the extended type and the second of the non-extended type, we have for the output count rate R the relation [1]

$$R = \frac{\rho}{(1-\alpha)\,x + e^{\alpha x}} , \tag{3}$$

where $\alpha = \tau_1/\tau_2 \leqslant 1$ and $x = \rho\,\tau_2$.

If written in the form of a power series, we find

$$\frac{\rho}{R} = 1 + x + \sum_{j=2}^{\infty} \frac{(\alpha\,x)^j}{j!} . \tag{4}$$

Comparison with the expression (2) valid for a generalized dead time shows that the condition R = R' requires that

$$\sum_{j=2}^{\infty} \frac{(\alpha\,x)^j}{j\,!} = \sum_{j=2}^{\infty} \frac{x^j}{j!}\,\theta^{j-1} ,$$

or likewise, for $J \geqslant 0$, that

$$\sum_{j=0}^{J} \frac{\alpha^{j+2}\,x^j}{(j+2)!} = \sum_{j=0}^{J} \frac{\theta^{j+1}}{(j+2)!}\,x^j . \tag{5}$$

We begin with the value J = 0 which leads immediately to the approximate, but very useful result

$$\theta \cong \theta_o = \alpha^2 . \tag{6}$$

Better approximations of θ, which will then also be a function of x, can be obtained step by step, for instance by means of the ansatz

$$\theta \cong \theta_J = \theta_o \left(1 + \sum_{j=1}^{J} a_j x^j\right) , \tag{7}$$

thus in the form of a power series in x.

For J = 1 equation (5) leads to

$$\alpha^2 + \frac{1}{3} \alpha^3 x = \theta + \frac{1}{3} \theta^2 x .$$

By putting now, according to (7), $\theta_1 = \theta_o(1 + a_1 x)$, we find by a development up to x

$$\alpha^2 + \frac{1}{3} \alpha^3 x \cong \alpha^2 + \alpha^2 a_1 x + \frac{1}{3} \alpha^4 x ,$$

from which one obtains readily

$$a_1 = \frac{1}{3} \alpha(1 - \alpha) . \tag{8}$$

For J = 2 we find from (5)

$$\alpha^2 + \frac{1}{3} \alpha^2 x + \frac{1}{12} \alpha^4 x^2 = \theta + \frac{1}{3} \theta^2 x + \frac{1}{12} \theta^3 x^2 .$$

As the previous choice of a_1 guarantees the equality of the terms on both sides up to x, we only have to consider those proportional to x^2.

Since now $\theta \cong \theta_2 = \alpha^2(1 + a_1 x + a_2 x^2)$ and knowing that

$$\begin{aligned} \theta_2 &\sim \alpha^2 a_2 x^2 , \\ \theta_2^2 x &\sim \alpha^4 2a_1 x^2 \quad \text{and} \\ \theta_2^3 x^2 &\sim \alpha^6 x^2 , \end{aligned}$$

we have for a_2 the condition

$$\frac{1}{12} \alpha^4 \cong \alpha^2 a_2 + \frac{1}{3} \alpha^4 2a_1 + \frac{1}{12} \alpha^6 .$$

With (8) this leads to

$$\begin{aligned} a_2 &= \frac{1}{\alpha^2} \left[\frac{1}{12} \alpha^4 - \frac{2}{3} \alpha^4 \frac{1}{3} \alpha (1-\alpha) - \frac{1}{12} \alpha^6\right] \\ &= \frac{1}{36} \alpha^2 (1 - \alpha) (3 - 5\alpha) . \end{aligned} \tag{9}$$

These successive approximations can be readily pursued and we have in this way found for J = 3 and 4 the coefficients

$$a_3 = \frac{1}{540}\,\alpha^3\,(1-\alpha)\,(9 - 41\alpha + 34\alpha^2)\quad , \tag{10}$$

$$a_4 = \frac{1}{6\,480}\,\alpha^4\,(1-\alpha)\,(18 - 174\alpha + 351\alpha^2 - 193\alpha^3)\quad . \tag{11}$$

When all these results are substituted into (7), we arrive at our final result for the parameter θ of the equivalent dead time, namely

$$\begin{aligned}\theta = \alpha^2 \Big\{1 + \frac{\alpha}{3}\,(1-\alpha)\,x\,\Big[1 &+ \frac{\alpha}{12}\,(3 - 5\alpha)\,x \\ &+ \frac{\alpha^2}{180}\,(9 - 41\alpha + 34\alpha^2)\,x^2 \\ &+ \frac{\alpha^3}{2\,160}\,(18 - 174\alpha + 351\alpha^2 - 193\alpha^3)\,x^3 + \ldots\Big]\Big\}\quad ,\end{aligned} \tag{12}$$

valid for a series arrangement of the type E-N.

4. The case N-E

For this arrangement of two dead times in series, the ouput count rate R is known [1] to be given by

$$R = \frac{\rho}{1 + \alpha x}\,e^{-(1-\alpha)x}\quad , \tag{13}$$

with the same notation as in (3). A series development in powers of x can be found in the following way. We first have

$$\frac{\rho}{R} = (1 + \alpha x)\,e^{(1-\alpha)x} = (1+\alpha x)\sum_{j=0}^{\infty}\frac{[(1-\alpha)x]^j}{j!}$$

$$= \sum_{j=0}^{\infty}\frac{(1-\alpha)^j\,x^j}{j!} + \alpha\sum_{j=0}^{\infty}\frac{(1-\alpha)^j\,x^{j+1}}{j!}\quad ,$$

and this may be brought into the form

$$\frac{\rho}{R} = 1 + \sum_{j=1}^{\infty}\frac{(1-\alpha)^j\,x^j}{j!} + \frac{\alpha}{1-\alpha}\sum_{j=1}^{\infty}\frac{j(1-\alpha)^j\,x^j}{j!}$$

$$= 1 + \sum_{j=0}^{\infty}\frac{(1-\alpha)^{j+1}\,x^{j+1}}{(j+1)!} + \frac{\alpha}{1-\alpha}\sum_{j=0}^{\infty}\frac{(j+1)\,(1-\alpha)^{j+1}\,x^{j+1}}{(j+1)!}\quad ,$$

thus finally

$$\frac{\rho}{R} = 1 + \sum_{j=0}^{\infty} \frac{x^{j+1}}{(j+1)!} (1-\alpha)^j (1 + j\alpha) \quad . \tag{14}$$

A comparison of this result with the output ρ/R' of a generalized dead time, which may be written in the form

$$\frac{\rho}{R'} = 1 + \sum_{j=0}^{\infty} \frac{\theta^j}{(j+1)!} x^{j+1} \quad , \tag{2'}$$

shows that the equality $R = R'$ requires that

$$\sum_{j=1}^{\infty} \frac{(1-\alpha)^j}{(j+1)!} (1+j\alpha)\, x^{j+1} = \sum_{j=1}^{\infty} \frac{\theta^j}{(j+1)!} x^{j+1} \quad . \tag{15}$$

Hence, if (14) and (2') should be identical up to order x^{J+2}, we must demand that (for $J \geqslant 0$)

$$\sum_{j=0}^{J} \frac{(1-\alpha)^{j+1}}{(j+2)!} \left[1 + (j+1)\alpha\right] x^j = \sum_{j=0}^{\infty} \frac{\theta^{j+1}}{(j+2)!} x^j \quad . \tag{16}$$

For $J = 0$, equation (16) leads readily to

$$\theta \cong \theta_o = 1 - \alpha^2 \quad . \tag{17}$$

Let us now look for better approximations to θ. This can be done in a way similar to that employed previously for the case E-N. This time it is convenient to use the ansatz

$$\theta \cong \theta_J = b_o + \sum_{j=1}^{J} b_j x^j \quad , \tag{18}$$

with $b_o = 1 - \alpha^2$.

For $J = 1$ one finds for θ the condition

$$1-\alpha^2 + \frac{2}{3!} (1-\alpha)^2 (1+2\alpha)\, x = \theta + \frac{2}{3!} \theta^2 x \ .$$

By putting for θ the approximation $\theta_1 = b_o + b_1 x$ and developping up to linear terms in x we find

$$1-\alpha^2 + \frac{1}{3} (1-\alpha)^2 (1+2\alpha)\, x \cong b_o + b_1 x + \frac{1}{3} b_o^2 x$$

$$= 1-\alpha^2 + b_1 x + \frac{1}{3} (1-\alpha^2)^2 x \quad ,$$

hence

$$b_1 = \frac{1}{3}(1-\alpha)^2(1-2\alpha) - \frac{1}{3}(1-\alpha^2)^2 = -\frac{1}{3}\alpha^2(1-\alpha)^2 . \tag{19}$$

For J = 2 we likewise put $\theta \cong \theta_2 = b_o + b_1 x + b_2 x^2$. For the determination of b_2 it is again sufficient to consider the terms proportional to x^2, for which we have

$$\begin{aligned} \theta_2 &\sim b_2 x^2 , \\ \theta_2^2 x &\sim 2 b_o b_1 x^2 \quad \text{and} \\ \theta_2^3 x^2 &\sim b_o^3 x^2 . \end{aligned}$$

Hence, the coefficient b_2 can be obtained from the condition

$$\frac{1}{12}(1-\alpha)^3(1+3\alpha)x^2 = \theta_2 + \frac{1}{3}\theta_2^2 x + \frac{1}{12}\theta_2^3 x^2 ,$$

thus
$$\frac{1}{12}(1-\alpha)^3(1+3\alpha) = b_2 + \frac{2}{3}b_o b_1 + \frac{1}{12}b_o^3 ,$$

from which we find, after some elementary rearrangements, that

$$\begin{aligned} b_2 &= \frac{1}{12}(1-\alpha)^3(1+3\alpha) + \frac{2}{9}\alpha^2(1-\alpha^2)(1-\alpha)^2 - \frac{1}{12}(1-\alpha^2)^3 \\ &= -\frac{1}{36}\alpha^2(1-\alpha)^3(1-5\alpha) . \end{aligned} \tag{20}$$

In an analogous way the coefficients of the corrective terms proportional to x^3 and x^4 have been determined as

$$b_3 = \frac{1}{540}\alpha^2(1-\alpha)^4(1+14\alpha-34\alpha^2) \quad \text{and} \tag{21}$$

$$b_4 = \frac{1}{6\,480}\alpha^2(1-\alpha)^5(1-3\alpha-123\alpha^2+193\alpha^3) . \tag{22}$$

Hence, for a series arrangement N-E of two dead times, the parameter θ of the equivalent generalized dead time is given up to fourth order in x by

$$\begin{aligned} \theta = (1-\alpha)\Big\{1+\alpha - \frac{\alpha^2}{3}(1-\alpha)\,x\,\Big[1 &+ \frac{(1-\alpha)}{12}(1-5\alpha)\,x \\ &- \frac{(1-\alpha)^2}{180}(1+14\alpha-34\alpha^2)\,x^2 \\ &- \frac{(1-\alpha)^3}{2\,160}(1-3\alpha-123\alpha^2+193\alpha^3)\,x^3 + \dots\Big]\Big\} . \end{aligned} \tag{23}$$

The formulae (12) and (23) are obviously the main result of this study. In both cases we do not know the general rule for the coefficients appearing in them. It is easy to verify that for the limiting cases $\alpha = 0$ and $\alpha = 1$ the value of θ resulting from (12) or (23) corresponds to the type of the second or the first dead time in the series arrangement, respectively, as one would expect.

If the numerical value of the equivalent dead time is used for an evaluation of the original count rate ρ, on the basis of the measured output rate R and the dead time τ_2, then a repetitive procedure will have to be adopted because the determination of the appropriate value of θ supposes - at least in principle - that x (and hence ρ) is known in advance. In reality, however, this is no severe obstacle, for in the lowest approximation θ is only a function of α, and in the higher ones the dependence on x is not critical (at least for $x \lesssim 2$). Hence, one or two numerical iterations should be sufficient.

References

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