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Some simple operations with series

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Abstract

For the inverse and the square root of a power series, recursion formulae can be readily found for the new coefficients. Similar, but more complicated, general relations are indicated which give the new coefficients in terms of the old ones alone.

1. Introduction

It often happens that, in dealing with a power-series expansion, a rearrangement is required which involves some elementary, but at times rather tedious, manipulations to be performed with the coefficients of the series in question.

This short note does not claim any mathematical rigour and problems of convergence remain untouched. Its only purpose is to assemble in a convenient way some useful expressions which otherwise (as has happened to us more than once when the search for the scattered notes of a previous computation became too time consuming) must always be derived from scratch - with the obvious danger of new errors. There is hardly anything which is really new in what follows. On the other hand, the corresponding material readily available from such an (otherwise) excellent reference work as [1] is surprisingly scarce and the expressions given there never go beyond fourth order, which is often insufficient.

Let us consider a power series of the normalized form

$$S \equiv 1 + \sum_{k=1}^{\infty} a_k x^k . \qquad (1)$$

The only two simple operations dealt with in what follows are the inversion and the square root of S.

2. The inversion

Let us start with the inversion (or reciprocal) of S, defined by

$$I = \frac{1}{S} \equiv 1 + \sum_{k=1}^{\infty} b_k x^k , \tag{2}$$

and the obvious task consists in establishing relations which allow us to evaluate the new coefficients b_k in terms of the original ones.

From

$$I \cdot S = 1$$

we have readily

$$1 + (b_1 + a_1)\, x + (b_2 + a_1 b_1 + a_2)\, x^2$$
$$+ (b_3 + a_1 b_2 + a_2 b_1 + a_3)\, x^3 + \ldots = 1 ,$$

and it thus follows that the condition that the coefficients of all the positive powers of x disappear can be stated as

$$\sum_{j=0}^{k} a_j\, b_{k-j} = 0, \quad \text{for every } k \geqslant 1 , \tag{3a}$$

if we put formally $a_o = b_o = 1$.

This is equivalent to the recurrence relation

$$b_k = - a_k - \sum_{j=1}^{k-1} a_j\, b_{k-j} . \tag{3b}$$

For most practical, and in particular all numerical, purposes relation (3) is all we need for determining the coefficients b_k.

In some cases, however, - as for instance for questions related to the propagation of errors - one may wish to dispose of a formula which gives b_k directly in terms of a_j, with $1 \leqslant j \leqslant k$.

The explicit determination of the coefficients b_k is just a matter of persistence. As the derivation is as obvious as it is tedious, we state directly the results for the coefficients of lowest order. They are

$$\begin{aligned}
b_1 &= -\, a_1 \;, \\
b_2 &= a_1^2 - a_2 \;, \\
b_3 &= -\, a_1^3 + 2\, a_1a_2 - a_3 \;, \\
b_4 &= a_1^4 - 3\, a_1^2a_2 + 2\, a_1a_3 + a_2^2 - a_4 \;, \\
b_5 &= -\, a_1^5 + 4\, a_1^3a_2 - 3\, a_1^2a_3 - 3\, a_1a_2^2 + 2\, a_1a_4 + 2\, a_2a_3 - a_5 \;, \\
b_6 &= a_1^6 - 5\, a_1^4a_2 + 4\, a_1^3a_3 + 6\, a_1^2a_2^2 - 3\, a_1^2a_4 \\
&\quad - 6\, a_1a_2a_3 + 2\, a_1a_5 - a_2^3 + 2\, a_2a_4 + a_3^2 - a_6 \;.
\end{aligned} \tag{4}$$

By looking carefully at (4) it is not difficult to detect the pattern underlying these expressions and hence to guess the general result. This is

$$b_k = \sum \prod_{j=1}^{k} a_j^{n_j} \, (-1)^n \, P(n_j) \;, \tag{5}$$

where the sum extends over all products of $a_j^{n_j}$ which fulfil the condition

$$\sum_j j \, n_j = k \;. \tag{6a}$$

Therefore, the number of terms in (5) is equal to the number of partitions of k, denoted by p(k) in [1].

The coefficients $P(n_j)$ indicate the number of permutations one can perform with the n elements a_j of which n_j are identical, hence

$$P(n_j) = \frac{n!}{\prod_j n_j!} \;, \tag{6b}$$

where we have made use of the abbreviation $n = \sum_j n_j$.

For checking purposes we then have also evaluated the two subsequent coefficients, i.e.

$$
\begin{aligned}
b_7 = &- a_1^7 + 6\ a_1^5 a_2 - 5\ a_1^4 a_3 - 10\ a_1^3 a_2^2 + 4\ a_1^3 a_4 + 12\ a_1^2 a_2 a_3 \\
&- 3\ a_1^2 a_5 + 4\ a_1 a_2^3 - 6\ a_1 a_2 a_4 - 3\ a_1 a_3^2 + 2\ a_1 a_6 \\
&- 3\ a_2^2 a_3 + 2\ a_2 a_5 + 2\ a_3 a_4 - a_7\ ,
\end{aligned}
\tag{4'}
$$

$$
\begin{aligned}
b_8 = &\ a_1^8 - 7\ a_1^6 a_2 + 6\ a_1^5 a_3 + 15\ a_1^4 a_2^2 - 5\ a_1^4 a_4 - 20\ a_1^3 a_2 a_3 \\
&+ 4\ a_1^3 a_5 - 10\ a_1^2 a_2^3 + 12\ a_1^2 a_2 a_4 + 6\ a_1^2 a_3^2 - 3\ a_1^2 a_6 \\
&+ 12\ a_1 a_2^2 a_3 - 6\ a_1 a_2 a_5 - 6\ a_1 a_3 a_4 + 2\ a_1 a_7 + 2\ a_2 a_6 \\
&- 3\ a_2^2 a_4 - 3\ a_2 a_3^2 + 2\ a_3 a_5 + a_2^4 + a_4^2 - a_8\ ,
\end{aligned}
$$

and they fully agree with what we expect according to (5).

Obviously, our heuristic "derivation" of (5) cannot pretend to be a proof. Yet, there is no real doubt that a more rigorous approach would lead to the same result.

3. The square root

Our second problem is how to take the square root of a power series. Hence, if S is still given by (1), we look for*

$$R = \sqrt{S} \equiv 1 + \sum_{k=1}^{\infty} b_k\, x^k\ , \tag{7}$$

where the new coefficients, called again b_k, remain to be determined. The condition now to be fulfilled is therefore $R^2 = S$. From

$$
\begin{aligned}
(1 + b_1 x + b_2 x^2 + b_3 x^3 + \dots)^2 = &\ 1 + 2\ b_1\ x + (b_1^2 + 2\ b_2)\ x^2 \\
&+ (2\ b_1 b_2 + 2\ b_3)\ x^3 + \dots
\end{aligned}
$$

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* Obviously − R is also a possible solution. Whether it is physically acceptable or not depends on the context in which it occurs. In what follows only + R will be considered.

we readily obtain the relations

$$a_1 = 2\,b_1\,,$$
$$a_2 = b_1^2 + 2\,b_2\,,$$
$$a_3 = 2\,b_1 b_2 + 2\,b_3\,, \tag{8}$$
$$a_4 = 2\,b_1 b_3 + b_2^2 + 2\,b_4\,,$$
$$a_5 = 2\,b_1 b_4 + 2\,b_2 b_3 + 2\,b_5\,, \quad \text{etc.},$$

which, when solved for b_k, lead to

$$b_1 = a_1/2\,,$$
$$b_2 = a_2/2 - b_1^2/2\,,$$
$$b_3 = a_3/2 - b_1 b_2\,, \tag{9}$$
$$b_4 = a_4/2 - b_1 b_3 - b_2^2/2\,,$$
$$b_5 = a_5/2 - b_1 b_4 - b_2 b_3\,, \quad \text{etc.}$$

The corresponding general recursion formula is

$$b_k = \frac{1}{2}\,(a_k - \sum_{j=1}^{k-1} b_j\, b_{k-j})\,. \tag{10a}$$

Since the integer $k \geqslant 1$ is either even or odd, it can always be written as $k = 2m$ or $k = 2m-1$, with m a natural number. Therefore b_k can also be obtained by the shorter summation

$$b_k = \frac{a_k}{2} - \sum_{j=1}^{m-1} b_j\, b_{k-j} - \frac{b_m^2}{2}\,\delta_{k,2m}\,, \tag{10b}$$

where δ is the Kronecker symbol.

Again, our problem is in principle solved by the recursion formula (10). However, it is also possible to derive explicit expressions for b_k which depend only on the coefficients a_j (with $1 \leqslant j \leqslant k$). Their evaluation becomes increasingly cumbersome; the first few of them are

$$
\begin{aligned}
b_1 &= \frac{1}{2} a_1 , \\
b_2 &= \frac{1}{8} (4 a_2 - a_1^2) , \\
b_3 &= \frac{1}{16} (8 a_3 - 4 a_1 a_2 + a_1^3) , \\
b_4 &= \frac{1}{128} (64 a_4 - 32 a_1 a_3 - 16 a_2^2 + 24 a_1^2 a_2 - 5 a_1^4) , \\
b_5 &= \frac{1}{256} (128 a_5 - 64 a_1 a_4 - 64 a_2 a_3 + 48 a_1^2 a_3 \\
&\qquad + 48 a_1 a_2^2 - 40 a_1^3 a_2 + 7 a_1^5) , \\
b_6 &= \frac{1}{1\,024} (512 a_6 - 256 a_1 a_5 - 256 a_2 a_4 + 192 a_1^2 a_4 - 128 a_3^2 + 384 a_1 a_2 a_3 \\
&\qquad - 160 a_1^3 a_3 + 64 a_2^3 - 240 a_1^2 a_2^2 + 140 a_1^4 a_2 - 21 a_1^6) , \quad \text{etc.}
\end{aligned}
\tag{11}
$$

This time it is less obvious to find the general law valid for these coefficients. After some trial and error it finally appeared that its form is rather similar to the one given in (5), namely

$$
b_k = \frac{a_k}{2} - \sum \prod_{j=1}^{k-1} a_j^{n_j} (-1)^n Q(n_j) , \tag{12a}
$$

where the sum again extends over all products of $a_j^{n_j}$ for which $\sum\limits_j j\, n_j = k$, with

$$
Q(n_j) = \frac{(2n-3)!!}{2^n \prod\limits_j n_j!} \tag{13}
$$

and $n = \sum\limits_j n_j$ as before. The double factorial is defined by

$$
m!! = m \cdot (m-2) \cdot (m-4) \cdot \ldots \cdot 4 \cdot 2
$$

or

$$
m!! = m \cdot (m-2) \cdot (m-4) \cdot \ldots \cdot 3 \cdot 1 ,
$$

depending on whether m is even or odd. Its appearance in this context is rather surprising.

If we agree to put $(-1)!! = 1$, the coefficients b_k may also be written in the form

$$b_k = \sum \prod_{j=1}^{k} a_j^{n_j} (-1)^{n-1} Q(n_j) \ . \tag{12b}$$

It is rather obvious that the general result (12) can be brought into a somewhat simpler form if we start, instead of from (1), from

$$S = 1 - 2 \sum_{k=1}^{\infty} \alpha_k x^k \tag{1'}$$

and look for

$$R = \sqrt{S} = 1 - \sum_{k=1}^{\infty} \beta_k x^k \ , \tag{7'}$$

for then the new coefficients are given by

$$\beta_k = \sum \prod_{j=1}^{k} \alpha_j^{n_j} Q'(n_j) \ , \tag{12'}$$

with

$$Q'(n_j) = \frac{(2n-3)!!}{\prod_j n_j!} \ . \tag{13'}$$

The sum still extends over all p(k) partitions. Finally, we note for those not familiar with double factorials that the expression $(2n-3)!!$ appearing in (13) or (13') allows the equivalent form

$$(2n-3)!! = \frac{(2n-2)!}{(n-1)!} \frac{1}{2^{n-1}} \ ,$$

although this brings no real simplification.

Again, two additional coefficients b_k have been (painstakingly) evaluated to see if they still agree with the conjecture given by (12). The findings are

$$b_7 = \frac{1}{2\ 048}(1\ 024\ a_7 - 512\ a_1a_6 - 512\ a_2a_5 + 384\ a_1^2a_5$$
$$- 512\ a_3a_4 - 320\ a_1^3a_4 + 768\ a_1a_2a_4 - 960\ a_1^2a_2a_3$$
$$+ 384\ a_2^2a_3 + 280\ a_1^4a_3 + 384\ a_1a_3^2 - 252\ a_1^5a_2$$
$$+ 560\ a_1^3a_2^2 - 320\ a_1a_2^3 + 33\ a_1^7)\ ,$$

(11')

$$b_8 = \frac{1}{32\ 768}(16\ 384\ a_8 + 4\ 480\ a_1^4a_4 - 4\ 032\ a_1^5a_3 - 10\ 080\ a_1^4a_2^2$$
$$+ 3\ 696\ a_1^6a_2 - 15\ 360\ a_1^2a_2a_4 + 17\ 920\ a_1^3a_2a_3$$
$$- 15\ 360\ a_1a_2^2a_3 + 12\ 288\ a_1a_3a_4 - 7\ 680\ a_1^2a_3^2 + 8\ 960\ a_1^2a_2^3$$
$$+ 6\ 144\ a_2^2a_4 + 12\ 288\ a_1a_2a_5 - 4\ 096\ a_4^2 - 1\ 280\ a_2^4$$
$$- 8\ 192\ a_1a_7 + 6\ 144\ a_1^2a_6 - 5\ 120\ a_1^3a_5 - 8\ 192\ a_2a_6$$
$$+ 6\ 144\ a_2a_3^2 - 8\ 192\ a_3a_5 - 429\ a_1^8)\ ,$$

and this confirms the reliability of (12).

4. Final remarks

Among the many operations one may be led to perform with a power series - the two cases treated above in some detail may just serve as an illustration - there is in particular also the "reversion problem". This consists, if we start from a series

$$y = x + a_2\,x^2 + a_3\,x^3 + \ldots\ , \tag{14a}$$

in finding the coefficients of the reversed power series

$$x = y + b_2\,y^2 + b_3\,y^3 + \ldots\ . \tag{14b}$$

Their general evaluation is quite a complicated matter which we do not want to treat here. In addition - and from a purely practical point of view - one can well be of the opinion that this is not a very urgent task either, since there exist excellent tabulations. By far the most extensive such listing can be found in an old publication by Van Orstrand [2]. However, he uses a slightly modified form of (14), namely

$$y = x\,(1 - \sum_{k=1}^{\infty} A_k\, x^k)$$

and (15)

$$x = y\,(1 + \sum_{k=1}^{\infty} B_k\, y^k)\ ,$$

because this eliminates the sign problem in the lengthy explicit expressions. The relation between the new and the previous coefficients is simply

$$A_k = -\,a_{k+1}$$

and (16)

$$B_k = b_{k+1}\ .$$

The tabulation in [2] gives all the coefficients B_k up to B_{12} (in the present notation) in terms of A_j, with $j \leqslant 1 \leqslant k$, and it has proved fully reliable in recent applications and extensive checks.

We gratefully acknowledge interesting discussions with Mme M. Boutillon on some problems related to this report, and C. Veyradier is to be thanked for his help in the evaluation of the coefficients listed in (4') and (11').

References

[1] "Handbook of Mathematical Functions" (ed. by M. Abramowitz and I.A. Stegun) NBS, AMS 55 (GPO, Washington, 1965[4])

[2] C.E. Van Orstrand: "Reversion of power series", Phil. Mag. (Series 6) 19, 366-376 (1910)

Note added in proof

M. Boutillon has just succeeded in proving our conjecture (5) by complete induction, and a similar attempt for (12) is well under way.

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