On the best measuring conditions for SESAM

by Jörg W. Müller

Bureau International des Poids et Mesures, F-92310 Sèvres

Abstract

A study is made concerning the experimental conditions for which the selective sampling method for measuring source activities reaches its highest precision. It turns out that the optimum range for activities is of the order of 10^5 Bq for the electronic equipment now in use. The information contained in the present report also allows one to determine the measuring time which is needed for obtaining a given statistical uncertainty under clearly specified measuring conditions.

One of the attractive features of the selective sampling (SESAM) method for absolute activity measurements of nuclides which undergo beta-gamma decay consists of the fact that it is also applicable at very high count rates, which is a region where the traditional coincidence method requires large and difficult corrections [1]. An immediate question thus arising is whether there exists a source activity for which this approach is best in the sense that a certain statistical precision is obtained in a minimum measuring time and, if so, where this will be located. It is the purpose of this report to outline a simple quantitative answer to this question.

We first have to determine which of the experimentally measured quantities are contributing in a significant way to the final uncertainty. Fortunately, this happens to be an easy task. The activity N_o of the radioactive source we want to measure can be written in the form

$$N_o = \rho_\beta / \varepsilon_\beta \, , \tag{1}$$

where ρ_β is the measured beta count rate (corrected for dead time, etc.) and ε_β is the counting efficiency of the proportional counter. ρ_β can usually be measured over such a long time interval that its statistical uncertainty is negligibly small*. Thus, we shall assume that in essence

* In fact, some problems do arise at very high count rates where the large dead-time corrections involved may become less accurate.

the uncertainty of N_o is due entirely to ε_β. In the SESAM method this quantity is given by

$$\varepsilon_\beta = 1 - g/G \ , \tag{2}$$

where g and G are arrival densities of gamma pulses in two distinct time regions, which in practice can be measured as average channel contents on a multichannel analyzer. For the normal situation where ε_β is not too far from unity, G is considerably larger than g. In addition, G is obtained by averaging over many more channels than is possible for g, which is restricted to the region of the "gap" in the time spectrum, caused by the extended dead time T. It is a realistic simplification, therefore, to assume that the precision attainable for ε_β, and thereby for N_o, is practically conditioned by g. The latter quantity, in turn, is obtained (Fig. 1) by

$$g = N_g/c \ , \tag{3}$$

where c is the number of channels considered (within the "gap") in which N_g gamma pulses have been registered during the given measuring time t. Since c is a known integer, the precision of g is determined by N_g, for which we can assume that Poisson statistics holds.

In a more quantitative way, the simple reasoning is therefore as follows. If s_y denotes the estimated standard deviation of a quantity y, and $r_y = s_y/y$ the corresponding relative uncertainty, we see from (2) that

$$s_{\varepsilon_\beta} = s_{1-\varepsilon_\beta} = (1 - \varepsilon_\beta)\ r_{1-\varepsilon_\beta} = (1 - \varepsilon_\beta)\ \sqrt{r_g^2 + r_G^2}$$

$$\cong (1 - \varepsilon_\beta)\ r_g \ ,$$

since $r_G \ll r_g$, as explained above. Assuming now that N_g follows Poisson statistics, we have $r_g = N_g^{-\frac{1}{2}}$ and therefore

$$r_{N_o} \cong r_{\varepsilon_\beta} = s_{\varepsilon_\beta}/\varepsilon_\beta \cong \left(\frac{1 - \varepsilon_\beta}{\varepsilon_\beta}\right) \frac{1}{\sqrt{N_g}} \ . \tag{4}$$

Hence, the precision of the measured activity N_o is highest, i.e. its relative uncertainty is minimum, if - for a given efficiency ε_β - the quantity N_g reaches a maximum.

In this way, the original problem of determining the best measuring conditions has been transformed into the simple question of evaluating the parameters which correspond to a maximum of N_g.

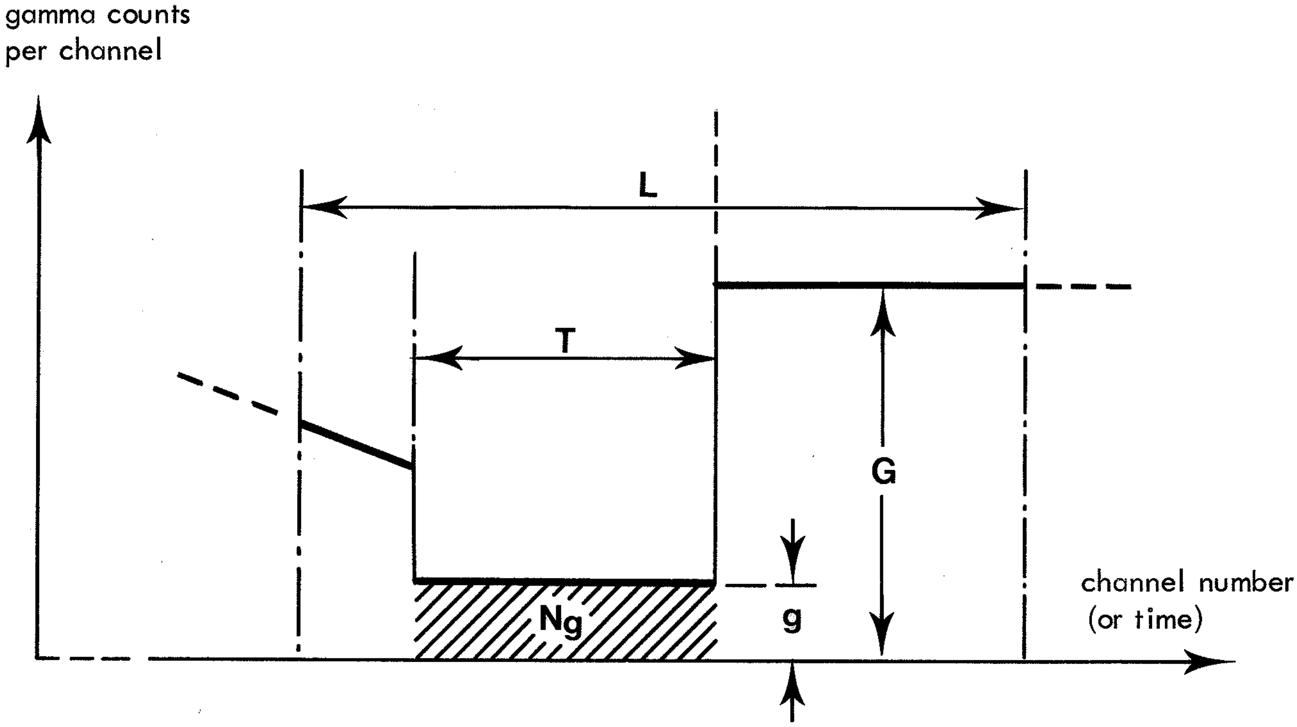

Fig. 1 - Schematic plot of the observed time spectrum of gamma pulses. The quantity of interest, N_g, corresponds to the hatched area.

For a given measuring time, the number of registered gamma counts N_g is clearly proportional to the number of "writing" cycles and to the average number of pulses P which arrive (within T) per cycle. In principle, a cycle can be initiated by any of the beta pulses* which have succeeded in passing the extended dead time T. For an original beta count rate ρ_β, the output rate R_β is given by

$$R_\beta = \rho_\beta \, e^{-\rho_\beta T} \; . \tag{5}$$

However, W must not be identified with R_β because the beta pulses arriving during a writing cycle cannot start a new cycle. This only becomes possible for the first event that occurs after the end of such a cycle. This situation corresponds exactly to a series arrangement of two dead times, where the first, of the extended type, has the value $\tau_1 = T$, whereas the second, non-extended, is given by the length of the writing cycle $\tau_2 = L$, for which we put

$$L = \kappa \, T \; . \tag{6}$$

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* We assume in what follows that the first beta pulse available will be used. This allows one to speed up the registration and is the mode now currently applied. Since the region before the "gap" is then distorted, the quantity G will be determined by means of the registered gamma pulses after the gap (Fig. 1).

Such a situation is schematically displayed in Fig. 2, and it is known that for a Poisson input r, the count rate observed after the two dead times is given by [2]

$$R = \frac{r}{(1 - \alpha)\ x + e^{\alpha x}} , \tag{7}$$

where $\alpha = \tau_1/\tau_2$ and $x = r\ \tau_2$. A derivation of this expression can be found in [3], for example.

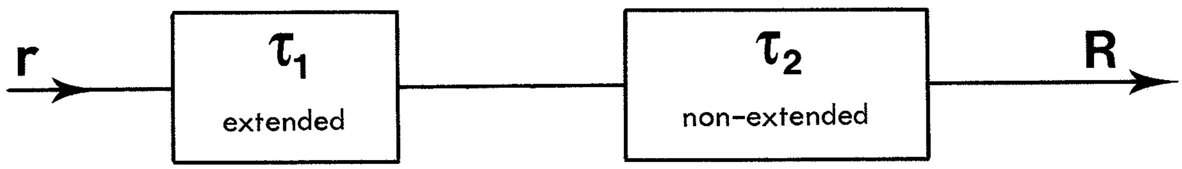

Fig. 2 - Scheme of the series arrangement of two dead times τ_1 and τ_2. The input and output count rates are r and R, respectively.

For the case we are considering here, this corresponds for the rate W of writing cycles, when the notation indicated above is used, to the relation

$$W = \frac{\rho_\beta}{(1 - \frac{1}{\kappa})\ \rho_\beta\ L + e^{\rho_\beta T}} = \frac{\rho_\beta}{(\kappa - 1)\ X_\beta + e^{X_\beta}} , \tag{8}$$

if we put $\rho_\beta\ T = X_\beta$.

As for the number of unpaired gamma pulses P registered per cycle within the "gap" one finds

$$P = \rho_\gamma\ (1 - \varepsilon_\beta)\ T = \frac{\varepsilon_\gamma\ (1 - \varepsilon_\beta)}{\varepsilon_\beta}\ \rho_\beta\ T . \tag{9}$$

We are now in a position to indicate an expression for the number of events N_g in a given measuring time t, which is

$$N_g = W\ P\ t = \frac{\varepsilon_\gamma\ (1 - \varepsilon_\beta)\ t}{\varepsilon_\beta}\ \frac{\rho_\beta^2\ T}{\rho_\beta\ (L - T) + e^{\rho_\beta T}} . \tag{10}$$

It may be useful to recall that by means of P. Brêonce's new "speed converter" (a detailed description of which is in preparation), the gamma pulses are registered during the "writing" cycle in a fast buffer-memory device. By repeated accumulation, all the arrival times corresponding to a very large number of cycles can thereby be stored in real time.
The transfer of this information to the multichannel analyzer is slow, but since it is so rarely performed, the time needed for this operation is negligible compared with the sum of the "writing" cycles. This is the reason why it has not been taken into account in the above analysis.

If, instead of the first beta pulse available, we should prefer to start a new cycle only with beta event number n, this would clearly reduce the writing cycles. For a start with pulse n, the quantity W, defined by (8) and appearing in (10), would have to be replaced by

$$W_n = \left(\frac{1}{W} + \frac{n-1}{R_\beta}\right)^{-1}, \qquad n = 1, 2, \ldots, \tag{8'}$$

with R_β given by (5). Obviously $W_1 = W$.

Since we are primarily interested in N_g as a function of the source activity N_o (which is proportional to ρ_β), as well as in its dependence on the length T of the cumulative dead time imposed, we can assume in what follows that ε_β, ε_γ and t are constants. The function to be studied is therefore

$$f(\rho,T) = \frac{\rho^2 T}{\rho(L - T) + e^{\rho T}}, \tag{11}$$

where ρ_β has been abbreviated to ρ for simplicity.

Let us first consider the situation where, for a given T, we look for the value of ρ which gives a maximum for N_g and hence the best precision for N_o, as explained before. For the partial derivative, an elementary calculation yields

$$\frac{\partial f}{\partial \rho} = \frac{\rho T}{\rho(L - T) + e^{\rho T}} \left[2 - \frac{\rho(L - T + T e^{\rho T})}{\rho(L - T) + e^{\rho T}}\right].$$

The condition for the maximum is therefore ($L/T = \kappa$)

$$\rho T (\kappa - 1 + e^{\rho T}) = 2 \left[(\kappa - 1) \rho T + e^{\rho T}\right],$$

that is

$$e^{X_\rho}(X_\rho - 2) = (\kappa - 1) X_\rho, \tag{12}$$

if we use the abbreviation $\rho_{opt} T = X_\rho$, where ρ_{opt} is the beta count rate which gives a maximum gap content N_g for a given value of T.

Equation (12) must be solved numerically; the result is given in graphical form in Fig. 3 for the range of practical interest. It can be seen that X_ρ does not depend in a critical way on the length L of the writing cycle.

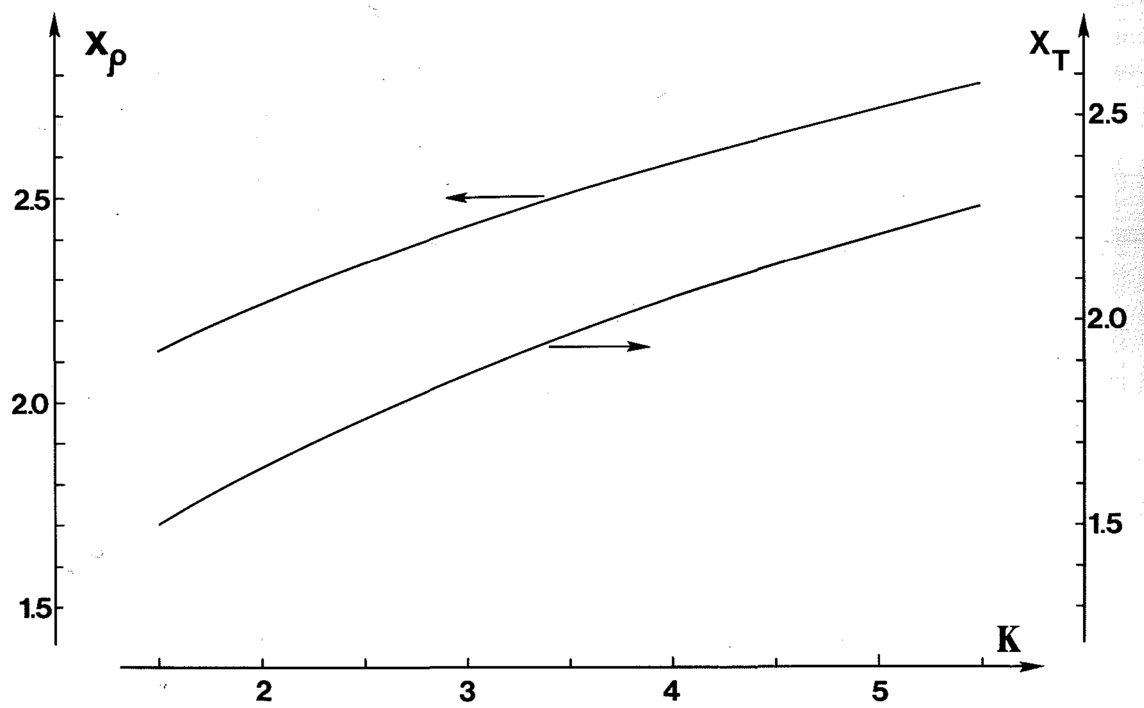

Fig. 3 - Graphical plot for the solutions of eq. (12) and eq. (13)'.

One may see that in the limiting case $\kappa = 1$ we have $X_\rho = 2$. This corresponds to the (unrealistic) situation where the whole writing cycle is performed during the dead-time period T and therefore causes no additional loss of time. However, such a measurement would only give g, instead of both G and g which are needed in (2) for the determination of ε_β.

Let us consider a practical numerical example. For T = 25 μs and L = 125 μs, for instance, one finds $X_\rho = 2.718$. This corresponds to a beta count rate $\rho \cong 109 \cdot 10^3\ s^{-1}$, thus to an optimum source activity N_o of about 120 kBq (assuming $\varepsilon_\beta \cong 0.9$). This confirms our previous statement that the SESAM method is particularly suited for measuring high activities.

While it is clearly useful to know for which activity the method has its highest sensitivity, it is perhaps even more interesting to dispose of information permitting one to compare the sensitivity over a wide range of source strengths. In view of the relation (4), we can simply use the quantity N_g for such a comparison, which is proportional to the function $f(\rho,T)$ given by (11). In Figures 4a and 4b we have plotted a number of curves which give N_g as a function of ρ for some representative parameters T and κ. They are all (quite arbitrarily) normalized in such a way that the maximum for T = 30 μs and $\kappa = 4$ reaches unity. We conclude from the shape of these curves that practical measurements can still be usefully performed at activities which correspond to several times the optimum value, in particular for the smaller dead times. Fig. 4 shows that efficient measurement of activities of 200 kBq and above requires T values in the range of 15 to 20 μs. On the other hand, the effect of κ on the total measuring time then tends to become negligible.

Within rather narrow limits, imposed by practical considerations (mainly related to instrumental dead times), the numerical value of the extended dead time can be freely chosen. For the sake of curiosity, at least, one may therefore also look for the optimum value of f when T is taken as a variable.

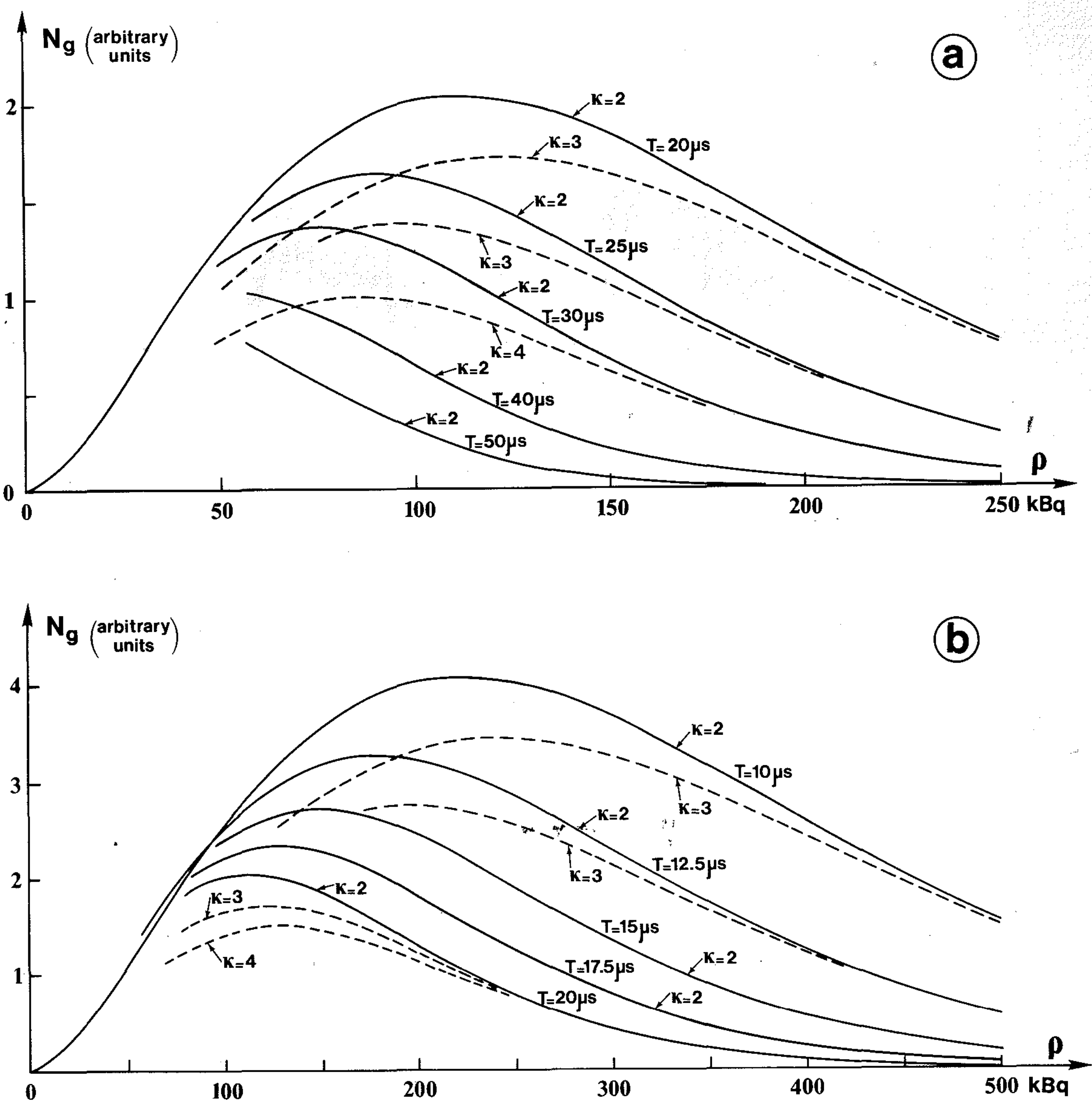

Fig. 4 - Graphical representation of the quantity N_g, as a function of the beta count rate ρ and for some values of the extended dead time T and the ratio $\kappa = L/T$. For the normalization, see text.
a) For $\rho \leqslant 250 \cdot 10^3\ s^{-1}$ and $T \geqslant 20\ \mu s$.
b) For $\rho \leqslant 500 \cdot 10^3\ s^{-1}$ and $T \leqslant 20\ \mu s$.

Starting again from (11), we find for the partial derivative

$$\frac{\partial f}{\partial T} = \frac{\rho^2}{\rho(L - T) + e^{\rho T}} \left[1 - \frac{\rho T \; (e^{\rho T} - 1)}{\rho(L - T) + e^{\rho T}}\right] .$$

This leads for the maximum of f to the condition

$$\rho T \; (e^{\rho T} - 1) = \rho(L - T) + e^{\rho T}$$

or

$$e^{X_T} \; (X_T - 1) = \kappa \; X_T \; , \qquad (13)$$

if we use again the variable $\kappa = L/T$ and denote $\rho \; T_{opt}$ by X_T .
A comparison of (13) with (12) shows that X_ρ and X_T are not identical, which means that the best measuring conditions depend on the quantity we choose as a "variable". However, this is not too surprising, for the function $f(\rho,T)$ given in (11) has no absolute maximum for $T > 0$ and ρ finite (cf. Fig. 3). For fixed values of either T or ρ we can only reach relative maxima for f and therefore N_g. We may note in passing that for T variable, the situation where the influence of L on the number of cycles is neglected corresponds to the limit $\kappa = 0$ and yields the value $X_T = 1$ (cf. Fig. 2).

For the results of some preliminary experimental tests see [4]; they are in good agreement with expectation. Numerical checks concerning the absolute values of N_g predicted by (10) have shown them to be very reliable. In practice, the simplest application of the results presented in this report may be as follows. For the given experimental conditions one first determines from (4) the value of N_g needed for attaining a prescribed precision r_{N_o} of the activity. This value, when substituted in (10), allows us to evaluate the corresponding measuring time t.

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References

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