Note on the decay correction required for a radionuclide ^{N}X in presence of its metastable state $^{N}X^{m}$.

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INTRODUCTION

When measuring a radioactive source composed of a radionuclide ^{N}X together with its metastable state $^{N}X^{m}$ decaying to the ground state, special attention should be given to the decay correction. Indeed the activity A of the radionuclide ^{N}X is the summation of the usual decaying activity A_D of the initial ^{N}X with the growing activity A_G of the ^{N}X coming from the disintegration of the metastable state to the ground state.

If t_0 is the time of production of the source (which is often not known by the user), the usual decay correction $A_D(t) = A_{D,0} \exp[-(t-t_0)/\tau]$ allows an easy calculation of the correction D for any time difference $\Delta t = t_2 - t_1$:

$$\begin{aligned} A_D(t_2) &= A_D(t_1) \exp[-(t_2-t_0)/\tau] \,/\, \exp[-(t_1-t_0)/\tau] \\ &= A_D(t_1) \exp[-\Delta t/\tau] = A_D(t_1)\, D(t_1, t_2) \qquad (1). \end{aligned}$$

On the other hand, the correction for the growing activity of ^{N}X coming from the metastable state is given by

$$A_G(t) = A_{m,0} \{\exp[-(t-t_0)/\tau_m] - \exp[-(t-t_0)/\tau]\}\, \tau_m / (\tau_m - \tau) \qquad (2),$$

where $A_{m,0}$ is the activity of the metastable state at the time t_0 [1]. The correction for a time difference Δt is then

$$\begin{aligned} A_G(t_2) &= A_G(t_1) \{\exp[-(t_2-t_0)/\tau_m] - \exp[-(t_2-t_0)/\tau]\} \,/\, \{\exp[-(t_1-t_0)/\tau_m] - \exp[-(t_1-t_0)/\tau]\} \\ &= A_G(t_1)\, G(t_0, t_1, t_2) \qquad (3). \end{aligned}$$

Depending on the respective half-lives τ and τ_m of both states, the equilibrium (constant activity ratio) between $^{N}X^{m}$ and the daughter ^{N}X may or may not be reached:

- In case of <u>τ smaller than τ_m</u> equilibrium is reached after a transition period. At equilibrium, the activity of the daughter ^{N}X can be deduced from the activity of the parent $^{N}X^{m}$ and, in consequence, the calculation of $G(t_0, t_1, t_2)$ is avoided. Indeed, it can be seen from (2) that when $(t-t_0) >> \tau$, the second term $\exp[-(t-t_0)/\tau]$ tends to zero faster than the first term and, in consequence, the daughter ^{N}X is decaying following the simple exponential law with the half-life of the metastable state.

- Obviously, in case of <u>τ larger than τ_m</u>, equilibrium cannot be reached and the time t_0 of the source production is needed to calculate $G(t_0, t_1, t_2)$. In the extreme case of <u>$\tau >> \tau_m$</u>, the metastable state rapidly decays to the ground state and the source, then composed of pure ^{N}X, decays normally following the simple exponential law.

In this short report, it is demonstrated that, perhaps surprisingly, the decay correction for the total activity $A = A_D + A_G$ is independent of t_0. No conditions on the respective half-lifes of ^{N}X and $^{N}X^{m}$ are imposed.

DEMONSTRATION

By definition, $\lambda = 1/\tau$ and $\lambda_{\mathrm{m}} = 1/\tau_{\mathrm{m}}$.
If $A_{\mathrm{m}}(t)$ is the activity of $^{N}X^{m}$ and the ratio $R(t) = A_{\mathrm{m}}(t) / A(t) = A_{\mathrm{m}}(t) / (A_{\mathrm{D}}(t) + A_{\mathrm{G}}(t))$, we have:

$$A_{\mathrm{D}}(t) = A_{\mathrm{D},0}\mathrm{e}^{-\lambda(t-t_0)}$$

$$A_{\mathrm{m}}(t) = A_{\mathrm{m},0}\mathrm{e}^{-\lambda_{\mathrm{m}}(t-t_0)}$$

$$A_{\mathrm{G}}(t) = BA_{\mathrm{m},0}\mathrm{e}^{-\lambda_{\mathrm{m}}(t-t_0)}\frac{\lambda}{\lambda-\lambda_{\mathrm{m}}}(1-\mathrm{e}^{(\lambda_{\mathrm{m}}-\lambda)(t-t_0)}) \qquad (4),$$

where the index 0 relates to the time t_0 of production of the source. The last equation is the general relation calculating the activity of a daughter radionuclide from the activity of the parent [1] and is equivalent to equation (2). The factor B corresponds to a possible branching ratio when the metastable state decays only partially to the ground state.

If the quantities A and R are known at a time t_1, the total activity A at any time t_2 is given by $A(t_2) = A_{\mathrm{D}}(t_2) + A_{\mathrm{G}}(t_2)$:

$$\begin{aligned}
A(t_2) &= A_{\mathrm{D},0}\mathrm{e}^{-\lambda(t_2-t_0)} + BA_{\mathrm{m},0}\mathrm{e}^{-\lambda_{\mathrm{m}}(t_2-t_0)}\frac{\lambda}{\lambda-\lambda_{\mathrm{m}}}(1-\mathrm{e}^{(\lambda_{\mathrm{m}}-\lambda)(t_2-t_0)}) \\
&= A_{\mathrm{D}}(t_1)\mathrm{e}^{-\lambda\Delta t} + BA_{\mathrm{m}}(t_1)\mathrm{e}^{-\lambda_{\mathrm{m}}\Delta t}\frac{\lambda}{\lambda-\lambda_{\mathrm{m}}}(1-\mathrm{e}^{(\lambda_{\mathrm{m}}-\lambda)\Delta t}\mathrm{e}^{(\lambda_{\mathrm{m}}-\lambda)(t_1-t_0)}) \\
&= (A(t_1) - A_{\mathrm{G}}(t_1))\mathrm{e}^{-\lambda\Delta t} \\
&\quad + BR(t_1)A(t_1)\frac{\lambda}{\lambda-\lambda_{\mathrm{m}}}\mathrm{e}^{-\lambda_{\mathrm{m}}\Delta t}(1-\mathrm{e}^{(\lambda_{\mathrm{m}}-\lambda)\Delta t}\mathrm{e}^{(\lambda_{\mathrm{m}}-\lambda)(t_1-t_0)}) \\
&= A(t_1)\mathrm{e}^{-\lambda\Delta t} - BR(t_1)A(t_1)\frac{\lambda}{\lambda-\lambda_{\mathrm{m}}}(1-\mathrm{e}^{(\lambda_{\mathrm{m}}-\lambda)(t_1-t_0)})\mathrm{e}^{-\lambda\Delta t} \\
&\quad + BR(t_1)A(t_1)\frac{\lambda}{\lambda-\lambda_{\mathrm{m}}}(\mathrm{e}^{-\lambda_{\mathrm{m}}\Delta t} - \mathrm{e}^{-\lambda\Delta t}\mathrm{e}^{(\lambda_{\mathrm{m}}-\lambda)(t_1-t_0)}) \\
&= A(t_1)\mathrm{e}^{-\lambda\Delta t} + BR(t_1)A(t_1)\frac{\lambda}{\lambda-\lambda_{\mathrm{m}}}(\mathrm{e}^{-\lambda_{\mathrm{m}}\Delta t} - \mathrm{e}^{-\lambda\Delta t}) \\
&= A(t_1)\left[\mathrm{e}^{-\lambda\Delta t} + BR(t_1)\frac{\lambda}{\lambda-\lambda_{\mathrm{m}}}(\mathrm{e}^{-\lambda_{\mathrm{m}}\Delta t} - \mathrm{e}^{-\lambda\Delta t})\right] \qquad (5) \\
&= A(t_1)C(t_1, t_2, R(t_1))
\end{aligned}$$

Expression (5) shows a first term corresponding to the simple case of pure ^{N}X decay. The second term may reach non-negligible values as shown in the numerical examples below. This term is independent of t_0, showing the advantage of using (5) to calculate the decay correction C for the total activity A, instead of calculating the decay for the initial ^{N}X only and evaluating a correction for the contribution of the $^{N}X^{m}$ decay.

In conclusion, when measuring a radioactive source composed of a radionuclide ^{N}X together with its metastable state $^{N}X^{m}$ decaying to the ground state with a branching ratio B, the decay correction for the *total* activity of ^{N}X is given by $C(t_1, t_2, R(t_1))$ for any time difference, i.e. whether equilibrium is reached or not. This expression is independent of the time of source production and is valid for any values of τ and τ_m. Finally, given that the measured quantity is usually the total activity of ^{N}X, i.e. A_D and A_G are generally not determined independently, equation (5) is particularly convenient.

NUMERICAL EXAMPLES

1. ^{133}Xe ($\lambda = 0.1322\ d^{-1}$) containing some $^{133}Xe^{m}$ ($\lambda_m = 0.3168\ d^{-1}$):

   if $R(t_1) = 10^{-3}$ and $\Delta t = -10$ d, the ratio of the second to the first term in (5) is equal to 3.8×10^{-3}.

2. ^{177}Lu ($\lambda = 0.1043\ d^{-1}$) containing some $^{177}Lu^{m}$ ($\lambda_m = 4.321\times10^{-3}\ d^{-1}$), $B = 0.217$:

   if $R(t_1) = 10^{-3}$ and $\Delta t = 20$ d, the ratio of the second to the first term in (5) is equal to 1.5×10^{-3}.

[1] K. S. Krane, *Introductory Nuclear Physics*, John Wiley & Sons ed., 1988.