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Asymptotic results for a modified renewal process and their application to counting distributions

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1. Introduction

A complete set of limiting values for the expectation and the variance of the number of registered pulses has been obtained in the past years for the practically important case of an original Poisson process which has been distorted by a dead time. A review of the results can be found e.g. in [1], where the references to some of the earlier work are included. From a systematic point of view, these individual and sometimes quite laborious derivations are not very satisfactory: one might prefer a mathematically more elegant and coherent approach where the different experimental conditions can be taken into account by an appropriate specification.

As a matter of fact, such a general treatment is indeed possible and the main mathematical tools have been available for many years. The approach to be followed is sketched in the pioneering work of Smith ([2, 3]) and all we need is a slight generalization of his results.

On the other hand, no effort will be made to arrive at a high degree of mathematical rigour. Anybody interested in matters of convergence region of s, uniqueness of a solution, minimum requirements for the moments, etc., should go back to Smith's papers, where the references to earlier work (in particular by Feller, Täcklind, Doob and Blackwell) are given. In view of the simple practical applications we have in mind, such questions seem to be of minor importance here.

For our purpose, any interval density $\varphi_k(t)$ describing the arrival time of event number k can be written in the form of the convolution

$$\varphi_k(t) = {}_0f(t) * \{f(t)\}^{*(k-1)} , \qquad k = 1, 2, 3, \ldots, \tag{1}$$

where f(t) denotes the interval density between registered counts, while ${}_0f(t)$ is the density for the first event after t = 0 which depends on the choice of the time origin. This general case is called a modified renewal process [4] and Table 1 shows how the initial density ${}_{o}f(t)$ should be chosen for the usual three experimental counting conditions.

Table 1 - Choice of ${}_{o}f(t)$ for the different counting processes, and notation for multiple densities*; $f(t)$ is the interarrival density and m_1 the corresponding mean value of t (see section 3).

Type of counting process		initial density ${}_{o}f(t)$	multiple densities $\varphi_k(t)$
ordinary process	(or)	$f(t)$	${}_{or}\varphi_k(t) = f_k(t)$
equilibrium process	(eq)	$\frac{1}{m_1}\int_t^\infty f(x)\,dx = g(t)$	${}_{eq}\varphi_k(t) = g_k(t)$
free counter process	(fr)	$f(t)\big\vert_{\tau=o} = h(t)$	${}_{fr}\varphi_k(t) = h_k(t)$

The cumulative interval densities are defined by

$$\Phi_k(t) = \int_o^t \varphi_k(x)\,dx \quad ; \tag{2}$$

in the more traditional terminology they correspond to $F_k(t)$, $G_k(t)$ and $H_k(t)$ for the processes of Table 1.

The notion of a modified process is thus seen to be sufficiently general to cover all the three practical cases one is normally interested in. Therefore, the basic theoretical developments have to be carried out only once and they can then be adapted to the special needs. This will give us general formulae valid for any ordinary, equilibrium or free counter renewal process. Finally, and in particular, these expressions can be applied to an original Poisson process which has been disturbed by a dead time of either the extended or the non-extended type. These final results will confirm the expressions found previously and are intended to illustrate the usefulness of this unified approach.

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* These correspondences are given for the convenience of the reader who is familiar with the usual notation; they will not be used explicitly in what follows.

2. Some preliminary relations

Let us denote by k the number of pulses counted in a time interval t. The quantities we are mainly interested in are the asymptotic mean value and the variance of k, for which we write $\hat{k}(t)$ and $\sigma_k^2(t)$, respectively. Asymptotic means here that the counting interval t is assumed to be very large compared to, say, the mean distance in time between subsequent counts.

For an integer variable like k it is often mathematically convenient to use factorial moments. Instead of determining directly the usual moments, we prefer to evaluate first the (ordinary) factorial moments $\Psi_r(t)$ which are defined by the expectation (cf. also Appendix of [5])

$$\Psi_r(t) \equiv E\left\{k_{(r)}\right\}, \tag{3}$$

where $k_{(r)} \equiv k\,(k-1)\,(k-2) \ldots (k-r+1)$ is a (falling) r-factorial*.

Both k and $r \leqslant k$ are positive integers; r is called the order of the moment.

It is readily verified that for the two moments of lowest order we have**

$$\Psi_1(t) = E\left\{k\right\} = \hat{k}(t) = m_1(t),$$

$$\Psi_2(t) = E\left\{k\,(k-1)\right\} = m_2(t) - m_1(t),$$

hence $\hat{k}(t) = \Psi_1(t)$ and (4a)

$$\sigma_k^2(t) = m_2(t) - m_1^2(t) = \Psi_2(t) + \Psi_1(t) - \Psi_1^2(t). \tag{4b}$$

The (total) probability density $d(t)$ for all events k, which is also known as renewal density, is in the general case given by

$$d(t) \equiv \sum_{i=1}^{\infty} \varphi_i(t). \tag{5}$$

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* This follows the usual practice. An alternative definition based on the less commonly used "rising" factorials $k^{(r)} \equiv k\,(k+1)\,(k+2) \ldots (k+r-1)$ would also be possible and $k^{(2)}$ has in fact been applied by Cox [4] in this context. However, this choice does not seem to provide any additional advantage.

** For the definition of the moments $m_r(t)$ see Section 3.

The corresponding cumulative density

$$D(t) \equiv \int_0^t d(x)\, dx \tag{6}$$

is sometimes also called renewal function.

In order to express the factorial moments defined in (3) in terms of the interval distributions, we need a general relation between the number k of arrivals in a given time interval t and the interval densities φ_k for the arrival of event number k. This is known to be given by (see e.g. [4])

$$\begin{aligned} \text{Prob}\ (k) &= \int_0^t \varphi_k(x)\, dx - \int_0^t \varphi_{k+1}(x)\, dx \\ &= \Phi_k(t) - \Phi_{k+1}(t)\ . \end{aligned} \tag{7}$$

Therefore, the general expression for the factorial moments can be written as

$$\begin{aligned} \Psi_r(t) &= \sum_{k=0}^{\infty} k_{(r)} \cdot \text{Prob}\ (k) \\ &= \sum_{k=r}^{\infty} k\,(k-1) \ldots (k-r+1) \left[\Phi_k(t) - \Phi_{k+1}(t) \right] . \end{aligned} \tag{8}$$

2. Evaluation of the first two factorial moments

For the determination of the expectation $\hat{k}(t)$ and the variance $\sigma_k^2(t)$ of the number of pulses k within the measuring time t, it is sufficient, according to (4), to evaluate $\Psi_1(t)$ and $\Psi_2(t)$. For $r = 1$ the relation (8) gives immediately

$$\Psi_1(t) = \sum_{k=1}^{\infty} k \left[\Phi_k(t) - \Phi_{k+1}(t) \right] = \sum_{k=1}^{\infty} \Phi_k(t)\ . \tag{9}$$

The second factorial moment is obtained likewise as

$$\begin{aligned} \Psi_2(t) &= \sum_{k=2}^{\infty} k\,(k-1) \left[\Phi_k(t) - \Phi_{k+1}(t) \right] \\ &= 2 \sum_{k=2}^{\infty} (k-1)\, \Phi_k(t)\ . \end{aligned} \tag{10}$$

As a consequence of (1) it is practical to use in what follows integral transforms where convolutions correspond to simple multiplications. Applying Laplace transforms in the usual way, we can therefore also write

$$\tilde{\Psi}_1(s) = \frac{1}{s} \sum_{k=1}^{\infty} \tilde{\varphi}_k(s) \qquad \text{and} \tag{9'}$$

$$\tilde{\Psi}_2(s) = \frac{2}{s} \sum_{k=2}^{\infty} (k-1) \cdot \tilde{\varphi}_k(s) \; . \tag{10'}$$

For the general case of a modified renewal process, the transformed interval density (1) is given by

$$\tilde{\varphi}_k(s) = {}_o\tilde{f}(s) \cdot \left[\tilde{f}(s)\right]^{k-1} \; . \tag{1'}$$

For the first two factorial moments we therefore have for a modified process (m)

$$\begin{aligned} {}_m\tilde{\Psi}_1(s) &= \frac{1}{s} \sum_{k=1}^{\infty} {}_o\tilde{f}(s) \left[\tilde{f}(s)\right]^{k-1} \\ &= \frac{1}{s}\, {}_o\tilde{f}(s) \sum_{i=0}^{\infty} \tilde{f}(s)^i = \frac{{}_o\tilde{f}(s)}{s\left[1 - \tilde{f}(s)\right]} \; , \end{aligned} \tag{11}$$

$$\begin{aligned} {}_m\tilde{\Psi}_2(s) &= \frac{2}{s} \sum_{k=2}^{\infty} (k-1)\, {}_o\tilde{f}(s) \left[\tilde{f}(s)\right]^{k-1} \\ &= \frac{2}{s}\, {}_o\tilde{f}(s) \sum_{i=1}^{\infty} i \cdot \tilde{f}(s)^i \\ &= \frac{2}{s} \cdot \frac{{}_o\tilde{f}(s) \cdot \tilde{f}(s)}{\left[1 - \tilde{f}(s)\right]^2} \; . \end{aligned} \tag{12}$$

3. Asymptotic values

As a result of the well-known Tauber theorem which states that

$$\lim_{t \to \infty} {}_m\Psi_r(t) = \lim_{s \to 0} \left[s \cdot {}_m\tilde{\Psi}_r(s) \right] , \tag{13}$$

limiting values of ${}_m\Psi_r(t)$ for large intervals t can be obtained by expanding the corresponding transform into a power series in s. For this purpose we recall that any transformed density $\tilde{f}(s)$ may be written in the form

$$\tilde{f}(s) \equiv E\{e^{-st}\} = \int_0^\infty \left[1 - st + \frac{1}{2!}(st)^2 - \frac{1}{3!}(st)^3 + \dots\right] \cdot f(t)\, dt$$

$$= 1 - s \cdot m_1(t) + \frac{1}{2} s^2 \cdot m_2(t) - \frac{1}{6} s^3 \cdot m_3(t) \pm \dots, \tag{14a}$$

where $m_r(t) \equiv E\{t^r\} = \int_0^\infty t^r \cdot f(t)\, dt$

are the ordinary moments of t (of order r).

For the initial density ${}_of(t)$ we put likewise

$${}_o\tilde{f}(s) = 1 - s \cdot {}_om_1(t) + \frac{1}{2} s^2 \cdot {}_om_2(t) \mp \dots \quad . \tag{14b}$$

By substituting (14) into (11) we find for a modified process

$${}_m\tilde{\Psi}_1(s) = \frac{1 - s \cdot {}_om_1 + \frac{1}{2} s^2 \cdot {}_om_2 \mp \dots}{s^2 (m_1 - \frac{1}{2} s \cdot m_2 \mp \dots)}$$

which, upon simple dividing, leads to

$${}_m\tilde{\Psi}_1(s) \cong \frac{1}{s^2} \frac{1}{m_1} + \frac{1}{s} \left(\frac{m_2 - 2\, {}_om_1\, m_1}{2\, m_1^2} \right) . \tag{15}$$

Since

$$\left[1 - \tilde{f}(s)\right]^2 = s^2 \left[m_1^2 - s \cdot m_1 m_2 + s^2 (\tfrac{1}{3} m_1 m_3 + \tfrac{1}{4} m_2^2) \mp \dots\right],$$

we get for ${}_m\tilde{\Psi}_2(s)$ in the same way, but after a somewhat lengthier division,

$${}_m\tilde{\Psi}_2(s) = \frac{2}{s^3} \cdot \frac{(1 - s\, {}_om_1 + \frac{1}{2} s^2\, {}_om_2 \mp \dots)(1 - sm_1 + \frac{1}{2} s^2 m_2 \mp \dots)}{m_1^2 - sm_1 m_2 + s^2 (\frac{1}{3} m_1 m_3 + \frac{1}{4} m_2^2) \mp \dots}$$

$$\cong \frac{2}{s^3} \cdot \frac{1}{m_1^2} - \frac{2}{s^2} \left(\frac{1}{m_1} + \frac{{}_om_1}{m_1^2} - \frac{m_2}{m_1^3} \right)$$

$$+ \frac{2}{s} \left(\frac{{}_om_1}{m_1} + \frac{{}_om_2 - m_2}{2\, m_1^2} - \frac{{}_om_1\, m_2}{m_1^3} - \frac{1}{3} \frac{m_3}{m_1^3} + \frac{3}{4} \frac{m_2^2}{m_1^4} \right) . \tag{16}$$

Both ${}_m\tilde{\Psi}_1(s)$ and ${}_m\tilde{\Psi}_2(s)$ have only been developped into power series up to terms proportional to $1/s$ and it is shown in the Appendix why this is sufficient.

Before the variance can be found, we have to evaluate the term ${}_m\Psi_1^2(t)$ (or its transform) which appears in (4b). However, this is no major obstacle. Since (15) shows that ${}_m\tilde{\Psi}_1(s)$ is of the form

$$ {}_m\tilde{\Psi}_1(s) \cong \frac{a}{s^2} + \frac{b}{s} , $$

this corresponds to the original

$$ {}_m\Psi_1(t) \cong a \cdot t + b , \qquad \text{for} \quad t \longrightarrow \infty , $$

and therefore

$$ {}_m\Psi_1^2(t) \cong a^2 t^2 + 2\,a\,b\,t + b^2 , $$

the transform of which is

$$ \mathcal{L}\left\{ {}_m\Psi_1^2(t) \right\} \cong \frac{2\,a^2}{s^3} + \frac{2\,a\,b}{s^2} + \frac{b^2}{s} , \qquad \text{for} \quad s \longrightarrow 0 . \tag{17} $$

In our case, due to (15), we have

$$ a = \frac{1}{m_1} \quad \text{and} \quad b = \frac{m_2 - 2\,{}_o m_1\, m_1}{2\,m_1^2} . $$

According to (4b) the transformed variance for a modified process is given by

$$ {}_m\tilde{\sigma}_k^2(s) = {}_m\tilde{\Psi}_2(s) + {}_m\tilde{\Psi}_1(s) - \mathcal{L}\left\{ {}_m\Psi_1^2(t) \right\} . $$

By substituting (15), (16) and (17) this can be brought, after some rearrangements, into the form

$$ {}_m\tilde{\sigma}_k^2(s) \cong \frac{1}{s^2}\left[\frac{m_2 - m_1^2}{m_1^3}\right] + \frac{1}{s}\left[\frac{{}_o m_1}{m_1} - \frac{{}_o m_1^2}{m_1^2} + \frac{{}_o m_2}{m_1^2} - \frac{1}{2}\frac{m_2}{m_1^2} - \frac{{}_o m_1\, m_2}{m_1^3} - \frac{2}{3}\frac{m_3}{m_1^3} + \frac{5}{4}\frac{m_2^2}{m_1^4}\right] . \tag{18} $$

The originals of (15) and (18) are the asymptotic expressions looked for, namely

$$ {}_m\hat{k}(t) \cong \frac{t}{m_1} - \frac{{}_o m_1}{m_1} + \frac{m_2}{2\, m_1^2} \qquad \text{and} \qquad (19) $$

$$ {}_m\sigma_k^2(t) \cong \left(\frac{m_2}{m_1^3} - \frac{1}{m_1}\right)\cdot t + \frac{1}{m_1}\left[{}_o m_1 - \frac{1}{m_1}\left({}_o m_1^2 + {}_o m_2 - \frac{m_2}{2}\right) - \frac{1}{m_1^2}\left({}_o m_1\, m_2 + \frac{2}{3}\, m_3\right) + \frac{5}{4}\,\frac{m_2^2}{m_1^3}\right]. \qquad (20) $$

Instead of using the ordinary moments $m_r(t)$ given in (14), the formulae can sometimes be simplified by adopting instead the central moments $\mu_r(t)$ which are defined by $E\left\{(t - m_1)^r\right\}$.

Denoting as is usual μ_2 by σ^2, the change can be made by the substitution

$$ \begin{aligned} m_2 &= \sigma^2 + m_1^2 , \\ m_3 &= \mu_3 + 3\, m_1 \sigma^2 + m_1^3 , \end{aligned} \qquad (21) $$

and likewise for the moments of the density ${}_o f(t)$. After some simple algebra we arrive at

$$ {}_m\hat{k}(t) \cong \frac{t}{m_1} + \frac{1}{2} - \frac{{}_o m_1}{m_1} + \frac{\sigma^2}{2\, m_1^2} \qquad \text{and} \qquad (22) $$

$$ {}_m\sigma_k^2(t) \cong \frac{\sigma^2}{m_1^3}\cdot t + \frac{1}{12} + \frac{{}_o\sigma^2}{m_1^2} - \frac{{}_o m_1\, \sigma^2}{m_1^3} - \frac{2}{3}\,\frac{\mu_3}{m_1^3} + \frac{5}{4}\,\frac{\sigma^4}{m_1^4} . \qquad (23) $$

The four asymptotic relations (19), (20), (22) and (23) are the main result of the present study; the subsequent sections deal with specific applications.

Possibly the above relations hint to a general rule implying that an asymptotic moment of order r for k is specified by the moments of the inter-arrival time which go to order r + 1 for the density f(t), but only to order r for ${}_o f(t)$. However, since this conjecture is for the moment only based on the cases with r = 1 and 2, it has to be considered with due caution.

4. Specification of the type of counting process

The general formulae which are valid for a modified renewal process will now be specialized to the usual three types of counting processes mentioned in Table 1.

a) Ordinary process

This is the simplest case. As ${}_of(t)$ is identical with $f(t)$, we can simply put ${}_om_1 = m_1$ and ${}_om_2 = m_2$ to obtain from (19) and (20) the corresponding expressions in terms of ordinary moments, i.e.

$$ {}_{or}\hat{k}(t) \cong \frac{t}{m_1} - 1 + \frac{m_2}{2\,m_1^2} \quad , \tag{24}$$

$$ {}_{or}\sigma_k^2(t) \cong \left(\frac{m_2}{m_1^3} - \frac{1}{m_1}\right)\cdot t - \frac{1}{2}\frac{m_2}{m_1^2} + \frac{5}{4}\frac{m_2^2}{m_1^4} - \frac{2}{3}\frac{m_3}{m_1^3} \quad . \tag{25}$$

These results are identical to formula (2.8) and theorem 5* in [2]. In terms of central moments, the relations are

$$ {}_{or}\hat{k}(t) \cong \frac{t}{m_1} - \frac{1}{2} + \frac{\sigma^2}{2\,m_1^2} \quad , \tag{26}$$

$$ {}_{or}\sigma_k^2(t) \cong \frac{\sigma^2}{m_1^3}\cdot t + \frac{1}{12} + \frac{5}{4}\frac{\sigma^4}{m_1^4} - \frac{2}{3}\frac{\mu_3}{m_1^3} \quad . \tag{27}$$

Equations (26) and (27) agree with the corresponding relations given in [4].

b) Equilibrium process

As noted in Table 1, the initial density ${}_of(t)$ has to be chosen for an equilibrium process such that

$$ {}_of(t) = \frac{1}{m_1}\int_t^{\infty} f(x)\,dx \quad , \tag{28a}$$

thus

$$ {}_o\tilde{f}(s) = \frac{1}{m_1\cdot s}\left[1 - \tilde{f}(s)\right] \quad . \tag{28b}$$

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* taking into account the notation introduced in (1.4)

By using the power expansion (14) and equating the corresponding coefficients of s^r, we find (see [6], eq. 40) the general relation

$$ {}_o m_r = \frac{m_{r+1}}{(r+1) \cdot m_1} , \qquad \text{for } r = 0, 1, 2, \ldots, \tag{29} $$

hence

$$ {}_o m_1 = \frac{m_2}{2\, m_1} , \qquad {}_o m_2 = \frac{m_3}{3\, m_1} $$

and

$$ {}_o\sigma^2 = \frac{1}{12\, m_1^2} (4\, m_1\, m_3 - 3\, m_2^2) . $$

Substitution of (29) into the general formulae (19) and (20) leads to

$$ {}_{eq}\hat{k}(t) = \frac{t}{m_1} , \tag{30} $$

$$ {}_{eq}\sigma_k^2(t) \cong \left(\frac{m_2}{m_1^3} - \frac{1}{m_1}\right) \cdot t + \frac{1}{2} \frac{m_2^2}{m_1^4} - \frac{1}{3} \frac{m_3}{m_1^3} , \tag{31} $$

where (30) is known to be an exact result.

In terms of central moments, the variance is also given by

$$ {}_{eq}\sigma_k^2(t) \cong \frac{\sigma^2}{m_1^3} \cdot t + \frac{1}{6} + \frac{1}{2} \frac{\sigma^4}{m_1^4} - \frac{1}{3} \frac{\mu_3}{m_1^3} . \tag{32} $$

Equation (32) agrees with the result given in [3] .

c) Free-counter process

For this case the results for the general modified process cannot be simplified. The moments ${}_o m_r$ or ${}_o\mu_r$ have to be taken as those belonging to the density f(t), but in the absence of a dead time. More explicit expressions are possible only if we specify the original counting process.

5. The case of an original Poisson process

After having specialized the general results to certain experimental counting conditions which depend on the way the process is started (choice of time origin), we make an important further restriction by assuming that the original process (i.e. before insertion of a dead time) was of the Poisson type, with count rate ρ . This will lead us to the now well-known asymptotic formulae which have been obtained before separately by repeated individual attacks. They are now derived as very special cases in a more general frame.

It is obviously of no relevance for the result whether one starts from the expression with ordinary or with central moments. We restrict ourselves to the latter ones as they are often a bit simpler; they are listed in Table 2 for convenience.

Table 2 - Moments for the interval density of a dead-time-distorted Poisson process. We use the abbreviations $x = \rho\tau$ and $y = e^x$, where ρ is the original count rate.

Moment	for dead time τ: non-extended	for dead time τ: extended	for free counter process
m_1	$\frac{1}{\rho}(1+x)$	$\frac{1}{\rho}y$	$\frac{1}{\rho}$
σ^2	$\frac{1}{\rho^2}$	$\frac{1}{\rho^2}y(y-2x)$	$\frac{1}{\rho^2}$
μ_3	$\frac{2}{\rho^3}$	$\frac{2}{\rho^3}y(y^2-3xy+\frac{3}{2}x^2)$	$\frac{2}{\rho^3}$

It may be practical to give the formulae for expectation and variance also in tabular form (Tables 3 and 4). They all have been known before; for some earlier references see [1].

Table 3 - Asymptotic expectation and variance for a Poisson process of rate ρ, distorted by a non-extended dead time τ. The abbreviations $x = \rho\tau$ and $\lambda = 1/(1+x)$ are used.

Process	$\hat{k}(t)$	$\sigma_k^2(t)$
ordinary	$\lambda(\rho t - x + \frac{1}{2}\lambda x^2)$	$\lambda^3\left[\rho t - \frac{1}{12}\lambda x(12-6x-4x^2-x^3)\right]$
equilibrium	$\lambda \cdot \rho t$	$\lambda^3\left[\rho t + \frac{1}{6}\lambda x^2(6+4x+x^2)\right]$
free counter	$\lambda(\rho t + \frac{1}{2}\lambda x^2)$	$\lambda^3\left[\rho t + \frac{1}{12}\lambda x^2(18+4x+x^2)\right]$

Table 4 - As in Table 3, but for an extended dead time τ . We put $e^x = y$.

Process	$\hat{k}(t)$	$\sigma_k^2(t)$
ordinary	$\frac{1}{y}(\rho t - x)$	$\frac{1}{y^2}\left[(y-2x)\,\rho t - x(y-3x)\right]$
equilibrium	$\frac{1}{y}\cdot\rho t$	$\frac{1}{y^2}\left[(y-2x)\,\rho t + x^2\right]$
free counter	$\frac{1}{y}(\rho t + y - 1 - x)$	$\frac{1}{y^2}\left[(y-2x)\,\rho t + 1 - y + x(2-y+3x)\right]$

Incidentally, it seems reasonable to think that such a more general framework as outlined in section 4 would also be the appropriate basis for a possible generalization of the study of counting processes to a more general type of dead time or to one which might be taken as a random quantity.

6. Asymptotic variance-to-mean ratios

It may be worthwhile to have a quick look at the asymptotic forms which the variance-to-mean ratio, defined by

$$V(t) \equiv \frac{\sigma_k^2(t)}{\hat{k}(t)}, \tag{33}$$

takes for the different cases considered above.

Some time ago, an example of the exact numerical behaviour of V has been presented for an equilibrium process and a non-extended dead time [7]*. These results revealed a fairly complicated structure which, however, could be very well confirmed by Monte Carlo simulations. More recently this ratio V has been studied very thoroughly by Libert [8] for the three processes and for both types of dead time. As little of real interest can be added to his results, we shall confine ourselves to a cursory treatment.

By using the general relations (22) and (23) it is obviously possible to give an expression for the asymptotic ratio V(t), and the same is true for the specific counting processes, using for instance the relations given in (26) and (27) or in (30) and (31). However, no further simplifications seem to be possible for these general expressions.

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* Note that in the legend of Fig. 1 the expressions "full line" and "broken line" should be interchanged.

It is only by assuming a specific original process, in particular a Poisson process, that some additional (although modest) developments become possible. Let us consider the various cases separately in what follows.

A - V(t) for a non-extended dead time

a) Ordinary process

By substituting the corresponding expressions from Table 3 into (33) we obtain

$$ {}_{or}V(t) \cong \lambda^2 \cdot \frac{1 - \vartheta \cdot \frac{\lambda}{12}(12 - 6x - 4x^2 - x^3)}{1 - \vartheta(1 - \frac{1}{2}\lambda x)} , \tag{34a} $$

where $\vartheta \equiv \frac{x}{\mu} = \frac{\tau}{t}$.

By a series development of the denominator we can also write (to first order in ϑ)

$$ {}_{or}V(t) \cong \lambda^2 \left[1 + \vartheta \cdot \frac{\lambda x}{12}(12 + 4x + x^2)\right] . \tag{34b} $$

b) Equilibrium process

This case is particularly simple. By use of Table 3 we find

$$ {}_{eq}V(t) \cong \frac{1}{\lambda\mu} \cdot \lambda^3 \left[\mu + \frac{1}{6}\lambda x^2 (6 + 4x + x^2)\right] $$

$$ = \lambda^2 \left[1 + \vartheta \cdot \frac{\lambda x}{6}(6 + 4x + x^2)\right] . \tag{35} $$

c) Free counter process

Since from Table 3 we have

$$ {}_{fr}\hat{k}^{-1}(t) \cong \frac{1}{\lambda}(\mu + \frac{1}{2}\lambda x^2)^{-1} \cong \frac{1}{\lambda\mu}(1 - \vartheta \cdot \frac{\lambda x}{2}) , $$

there results for this case

$$ {}_{fr}V(t) \cong \lambda^2 \left[1 + \vartheta \cdot \frac{\lambda x}{12}(18 + 4x + x^2)\right] \cdot (1 - \vartheta \cdot \frac{\lambda x}{12} \cdot 6) $$

$$ \cong \lambda^2 \left[1 + \vartheta \cdot \frac{\lambda x}{12}(12 + 4x + x^2)\right] , \tag{36} $$

which coincides with the corresponding asymptotic value (34b) for the ordinary process.

Hence, neglecting the small difference in the term which is proportional to $x^2 \vartheta$, we arrive at a common asymptotic approximation for non-extended (n) dead times, namely

$$V_n(t) \cong \lambda^2 (1 + \vartheta \cdot \lambda x) , \tag{37}$$

which is supposed to be adequate for most practical cases.

B - V(t) for an extended dead time

a) Ordinary process

From the values given in Table 4 we can form the ratio

$${}_{or}V(t) \cong \frac{1}{y}\left[\frac{y - 2x - \vartheta (y - 3x)}{1 - \vartheta}\right] = 1 - \frac{x (2 - 3 \vartheta)}{y (1 - \vartheta)} , \tag{38a}$$

which is rigorous for $\vartheta \leqslant 0.5$ (i.e. $t \geqslant 2\tau$).

By a series development of the denominator, we obtain the simple approximate form

$${}_{or}V(t) \cong 1 - \frac{x}{y} (2 - \vartheta) . \tag{38b}$$

b) Equilibrium process

In this case the ratio V becomes

$${}_{eq}V(t) \cong \frac{1}{y}\left[(y - 2x) + \frac{x^2}{\mu}\right] = 1 - \frac{x}{y} (2 - \vartheta) , \tag{39}$$

and the formula is exact for $\vartheta \leqslant 1$.

c) Free counter process

After some rearrangements, V can here be brought into the form

$${}_{fr}V(t) \cong 1 - \frac{1}{y}\left[\frac{2x (\mu - 1) + y^2 - 3x^2 - 1}{\mu + y - 1 - x}\right] , \tag{40a}$$

a result which is rigorous for $t \geqslant 2\tau$.

An equivalent form including ϑ instead of μ, exact for $\vartheta \leqslant 0.5$, reads

$${}_{fr}V(t) \cong 1 - \frac{1}{y}\left[\frac{2x^2 + \vartheta (y^2 - 1 - 2x - 3x^2)}{x + \vartheta (y - 1 - x)}\right] . \tag{40b}$$

A comparison of the results given for an extended dead time with those indicated in [8] shows that they are identical or equivalent.

As a common asymptotic approximation for an extended (e) dead time, the expression

$$V_e(t) \cong 1 - \frac{x}{y}(2 - \vartheta) \tag{41}$$

can be used.

It has been noted by Libert that in general a much smoother behaviour of V will be obtained if the empirical mean value $\hat{k}(t)$ is used as the independent variable instead of the original expectation $\mu = \rho t$. In a few cases – especially for an extended dead time – exact relations for V can then be obtained (for $\vartheta \leq 0.5$ or 1). Thus, starting from (38a), which may be written as

$${}_{or}V(t) = 1 - \frac{1}{y}(\mu - x) \cdot \frac{x\,(2 - 3\,\vartheta)}{(\mu - x)\cdot(1 - \vartheta)} ,$$

and substituting

$$\hat{k} = \frac{1}{y}(\mu - x) ,$$

we arrive at

$${}_{or}V(t) = 1 - \hat{k} \cdot \frac{\vartheta(2 - 3\vartheta)}{(1 - \vartheta)^2} \; . \tag{42}$$

Likewise one gets for the equilibrium case from (39)

$${}_{eq}V(t) = 1 - \frac{\mu}{y}\,\frac{x}{\mu}\,(2 - \vartheta) = 1 - \hat{k}\cdot\vartheta(2 - \vartheta) \, . \tag{43}$$

Both (42) and (43), where V is a linear function of $\hat{k}$, have already been given in [8] .

Finally, we may state that for the non-extended case, since $\lambda \cong 1 - \hat{k}\,\vartheta$ and $\lambda x = 1 - \lambda$, (37) can also be written as

$$V_n(t) \cong (1 - \hat{k}\vartheta)^2 \cdot (1 + \hat{k}\,\vartheta^2) \, . \tag{44}$$

Similarly, it follows from (41) for an extended dead time, where $\hat{k} \cong \mu/y$, that

$$V_e(t) \cong 1 - \vartheta(2 - \vartheta)\,\hat{k} \, . \tag{45}$$

Hence, a simple approximate form, valid for any type of dead time, is given by

$$V(t) \cong 1 - 2\vartheta \cdot \hat{k} \, . \tag{46}$$

The variance-to-mean ratio is a practical means of describing the deviation of the experimental statistics from that of a Poisson process. However, it is necessarily an incomplete characteristic and in particular it should not be considered as a standard method for determining the dead time involved. Besides, there will often exist additional contributions to the variance (as e.g. fluctuations in the detection efficiency) which are difficult to measure separately. In any case, such a result would need confirmation by an independent method.

APPENDIX

On the series development of Laplace transforms

The question has recently been asked by one of our correspondents why the formal power series developments of a Laplace transform (as they have been used here e.g. in section 3) are always stopped at terms which are proportional to $1/s$. As this problem may also come up to the reader, we shall try to answer it here.

For this purpose let us consider a function $F(t)$, the transform of which presents itself in the form of a power series in s, thus

$$\tilde{F}(s) \equiv \mathcal{L}\{F(t)\} = \sum_i c_i \cdot s^i = \sum_{n=1}^{n'} a_n \cdot \frac{1}{s^n} + \sum_{m=1}^{m'} b_m \cdot s^{m-1} , \qquad \text{(A1)}$$

where the coefficients c_i, a_n and b_m are independent of s, and with n' and $m' \geqslant 1$.

For the usual one-sided definition of the transform of a given function $f(t)$, namely

$$\tilde{f}(s) \equiv \mathcal{L}\{f(t)\} \equiv \int_0^\infty f(t) \cdot e^{-st}\, dt , \qquad \text{(A2)}$$

where the original variable t is integrated over positive values, the well-known rule for differentiation (of order m) of a function $f(t)$ says that

$$\mathcal{L}\{f^{(m)}(t)\} = s^m \cdot \tilde{f}(s) - \sum_{i=1}^{m} s^{m-i} \cdot f^{(i-1)}(0) , \qquad \text{(A3)}$$

with $f^{(0)}(t) \equiv f(t)$ and $m = 0, 1, 2, \ldots$

An annoying problem seems to arise here from the need to know in (A3) the derivatives of $f(t)$ at the origin. Happily, this can be circumvented in the following way. If $f(t)$ vanishes for negative arguments (which is our case since t is a positive time interval), the integration in the defining transform can also be performed from $-\infty$ to $+\infty$. In other words, we may interpret our transformed functions as two-sided (or bilateral) Laplace transforms, i.e.

$$\tilde{f}(s) \equiv \mathcal{L}_{II}\{f(t)\} \equiv \int_{-\infty}^{\infty} f(t) \cdot e^{-st}\, dt \,. \tag{A4}$$

In this case, however, the rule for differentiation is much simpler (as may be found in any good textbook) and reads

$$\mathcal{L}_{II}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m \cdot \mathcal{L}_{II}\{f(t)\} = s^m \cdot \tilde{f}(s) \,, \tag{A5}$$

or likewise

$$\mathcal{L}_{II}^{-1}\{s^m \cdot \tilde{f}(s)\} = \frac{d^m f(t)}{dt^m} \,. \tag{A6}$$

Let us now make the special choice $f(t) = U(t)$, hence $\tilde{f}(s) = 1/s$. In order to determine the original of $\tilde{F}(s)$, we look at the second sum in (A1). For any term $m \geqslant 1$ we get, by comparing with (A6),

$$\mathcal{L}_{II}^{-1}\left\{b_m\, s^m \cdot \frac{1}{s}\right\} = b_m \cdot \frac{d^m}{dt^m} U(t) = 0 \,, \qquad \text{for } t \neq 0 \,. \tag{A7}$$

This shows that in the series development of $\tilde{F}(s)$ in (A1) it will be sufficient to consider the first sum over n, since all the originals corresponding to the terms of the sum over m vanish according to (A7). The highest power in s to be considered is thus s^{-1}. Therefore, the original function corresponding to (A1) is given by

$$F(t) = \sum_{n=1}^{n'} \frac{a_n}{(n-1)!} \cdot t^{n-1} \,, \qquad \text{for } t > 0.$$

In particular, $F(t)$ has no terms proportional to t^{-1} or t^{-2}, etc.

This is only apparently different if algebraic operations have first to be performed with the transformed quantities (e.g. multiplication): the s^{-1} rule then has to be respected for the final transformed result of which the original is determined by a term-by-term inversion.

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