Basic of time and frequency analysis and steering

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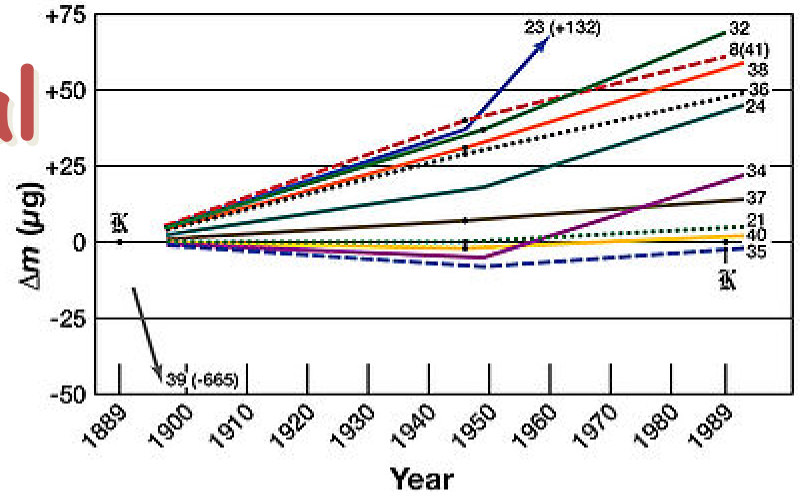
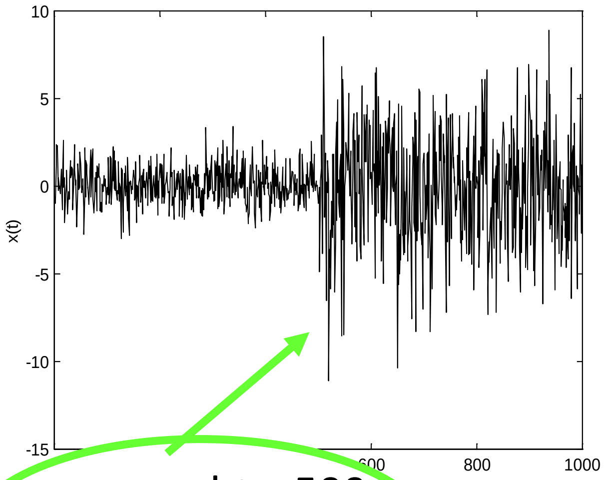
In time and frequency metrology we repeat the measurements and we look for their «stability»

Repeated measurements reveal always something

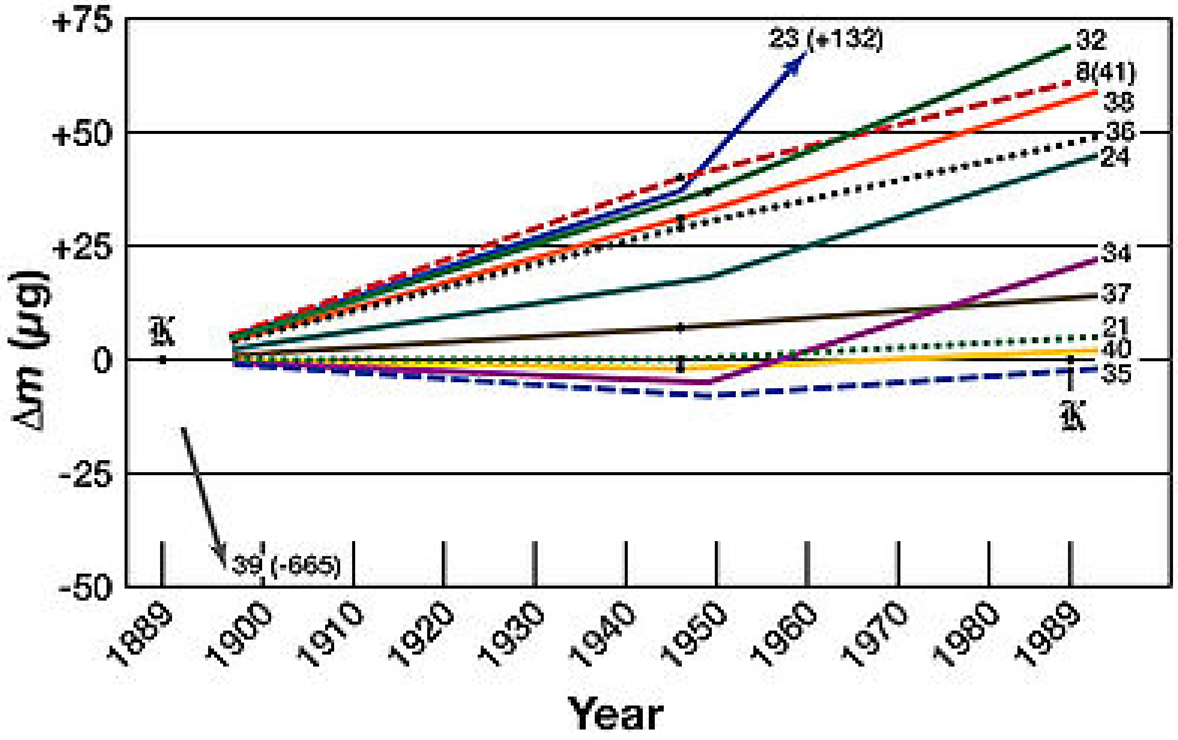


- *stationarity*
- *aging*
- *slow drift*
- *abrupt change*
- *failure*



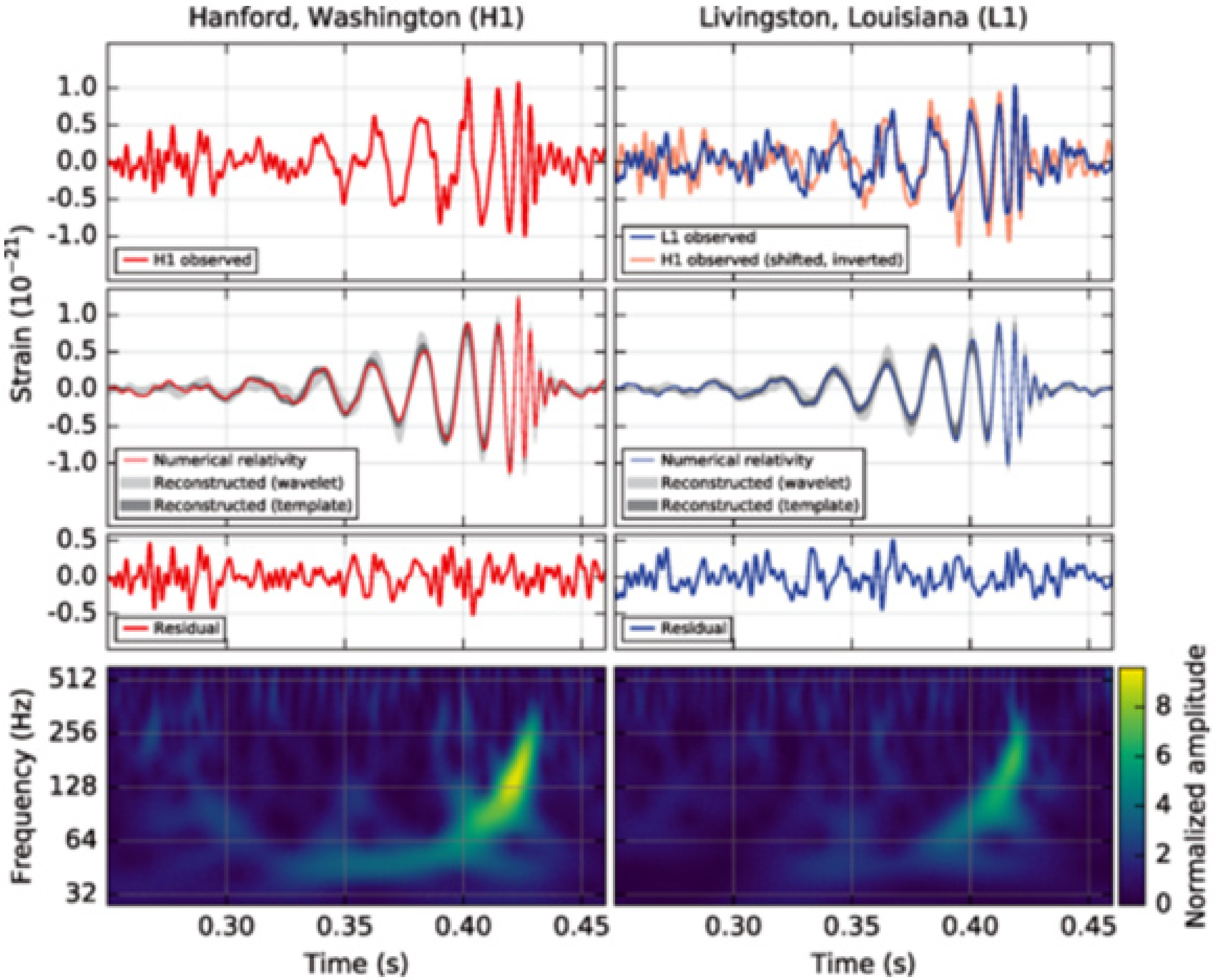
around t = 500 the noise increases

Changes may cause disturbance: the mass of the International Prototype Kilogram



Masses of the ensemble of prototypes have been slowly diverging from each other.

Or may reveal new physics



LIGO Sees First Gravitational Waves

What is changing?

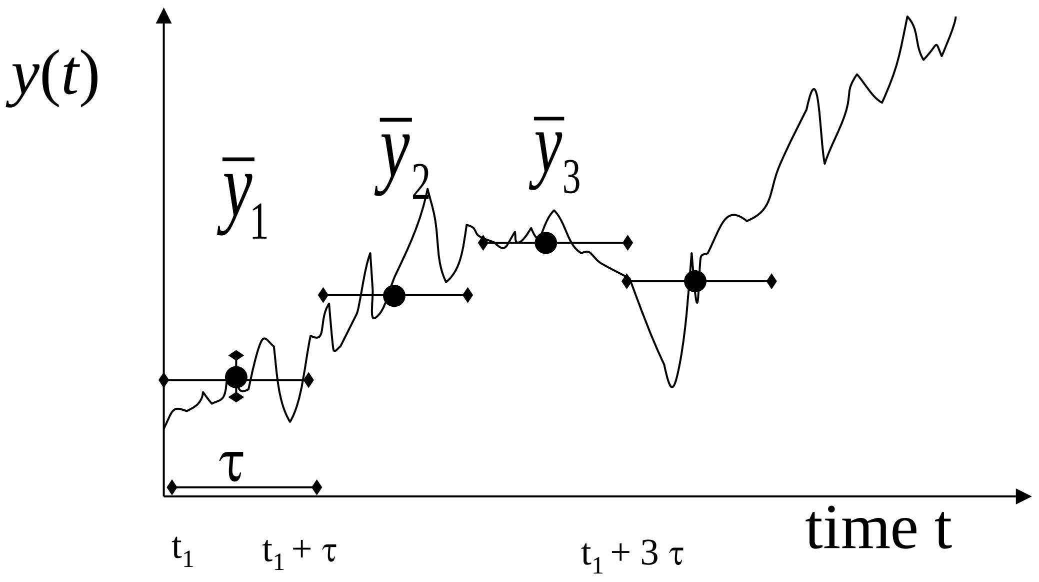
- Sometime the physical quantity (the measurand) is varying
- Sometime it is the measuring system

- The value itself depends on time
- or just the noise

- The uncertainty may vary with time, even if it is perfectly estimated now, it may not be valid tomorrow

Repeating measurements

The uncertainty on the single measure can be evaluated according to the GUM: Guide for Uncertainty in Measurement



In same cases, instability is more impacting than single measure uncertainty

Stability

Accuracy

Con

Cl

Stable or accurate?

DR. ALAN HOWARD, NE LABS, SCOTLAND

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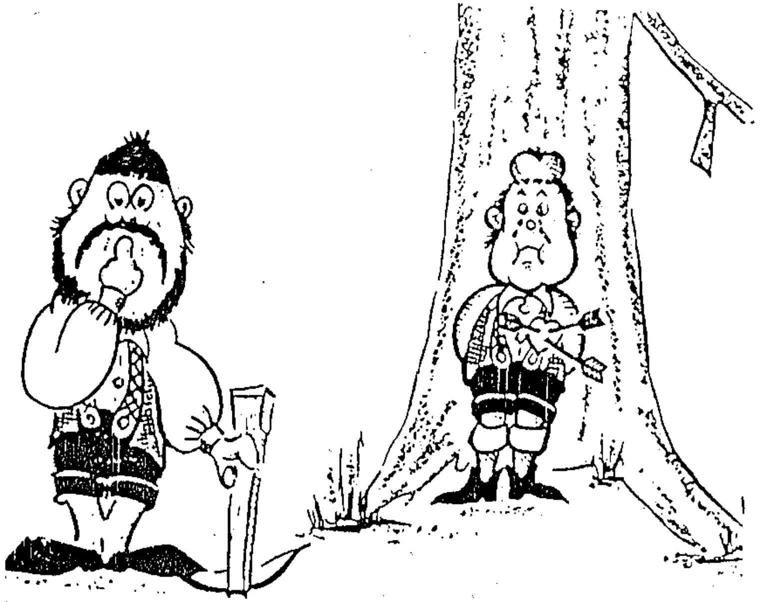

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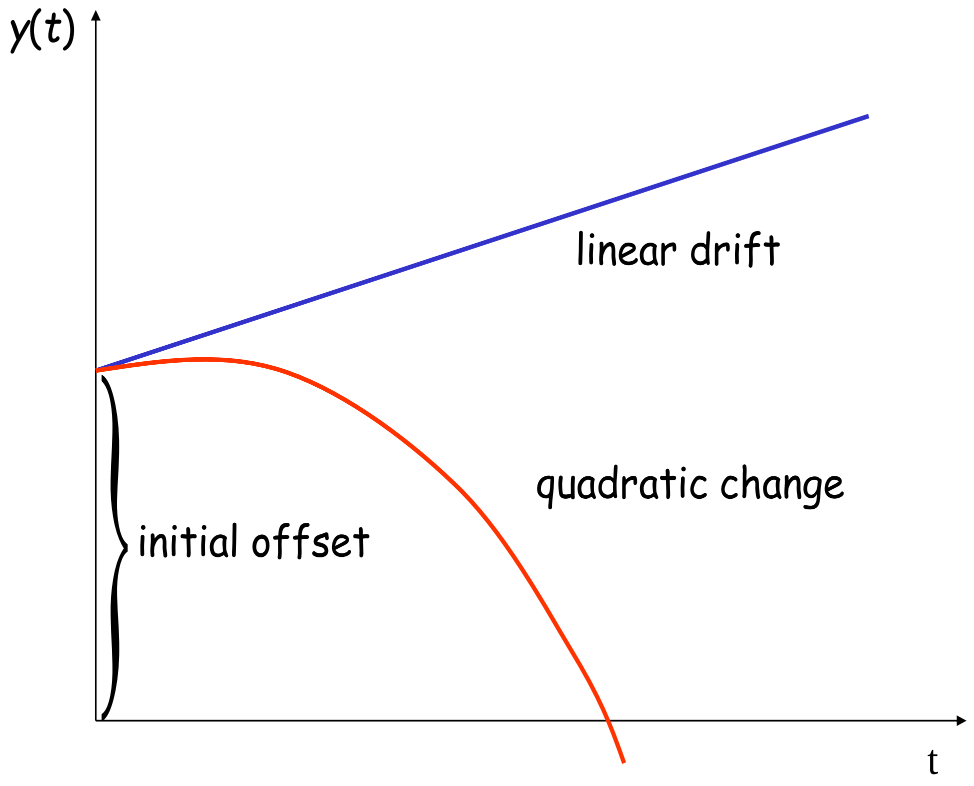
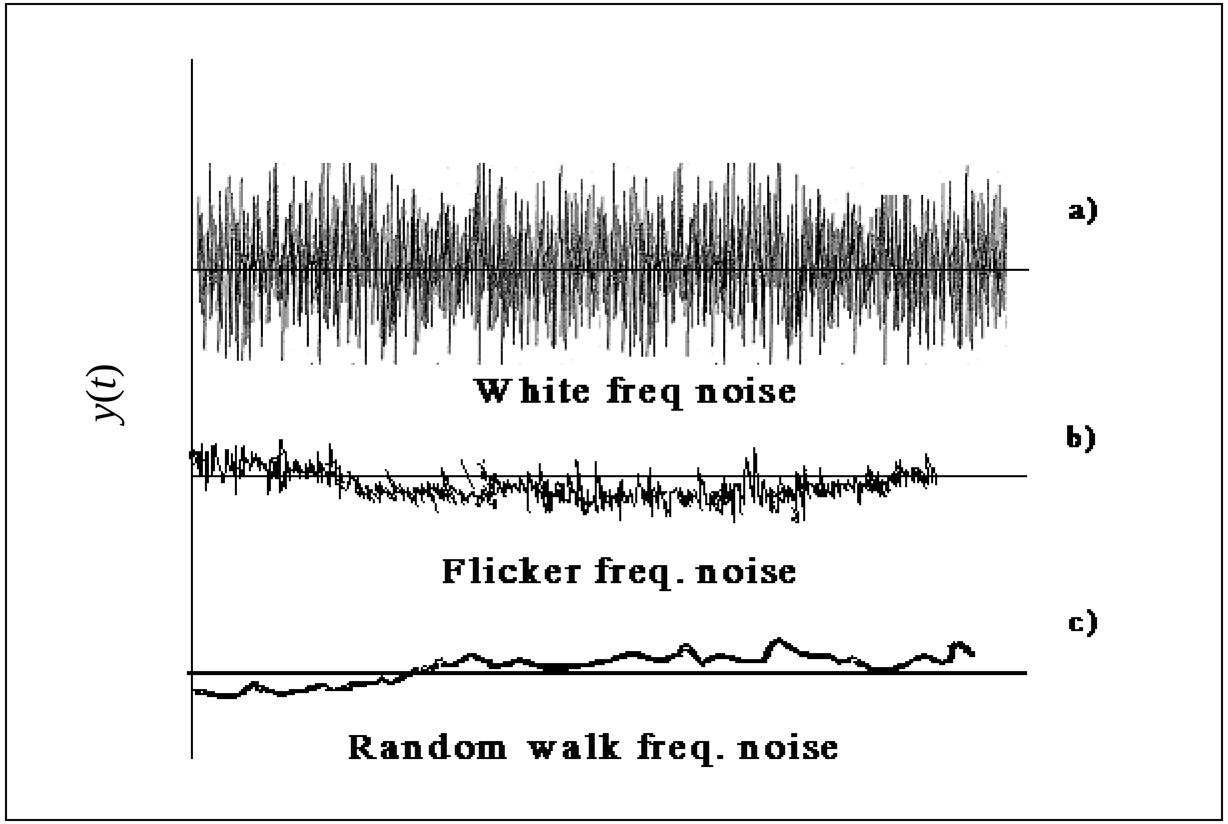
Time series of measures may contain polynomial component to be treated by least square estimation techniques (batch, recursive…)



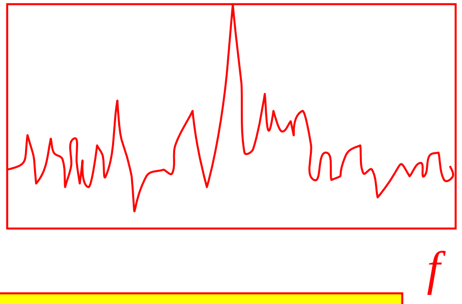
but stochastic components are also present and sometime dominant



At each instant the measure y(t) is a random variable

Appropriate mathematical tools for describing the stochastic noises:

- Spectral analysis of noise

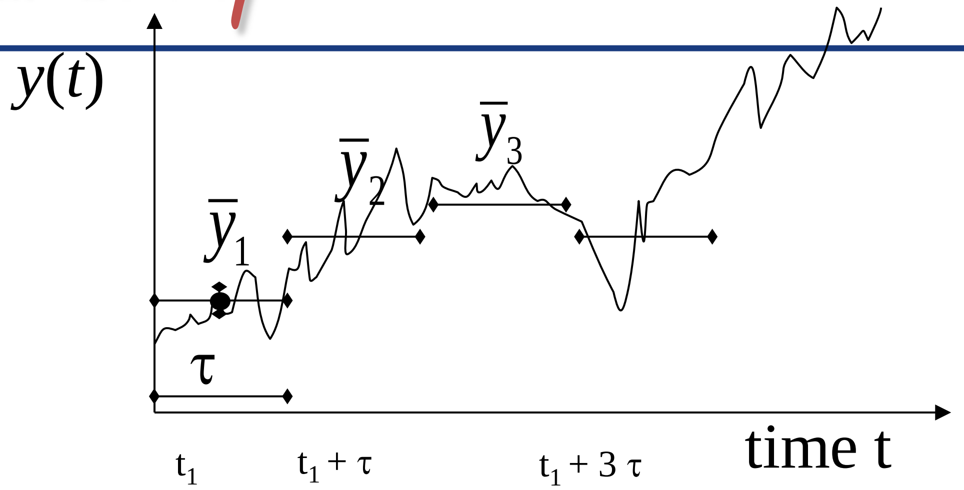


decomposition in elementary frequencies

- Filtered variances such as the Allan variance

dispersion of the average frequency values

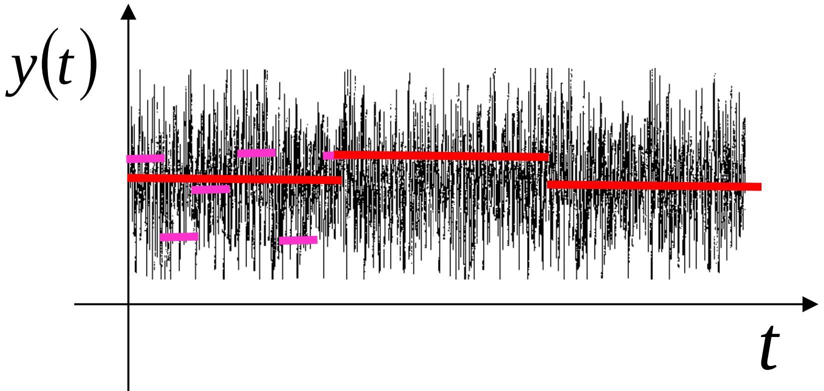
Instability - dispersion - variability



The variance is the measure of dispersion

$$\sigma^2(\tau) = \frac{\sum_{i=1}^{N} (\bar{y}_i - \mu)^2}{N}$$

First case of typical noisy behaviour (short term noise):



If the "noise" is stationary, the estimation of the variance converges as N grows

For longer observation interval τ, the variability diminishes
The "noise" is white and

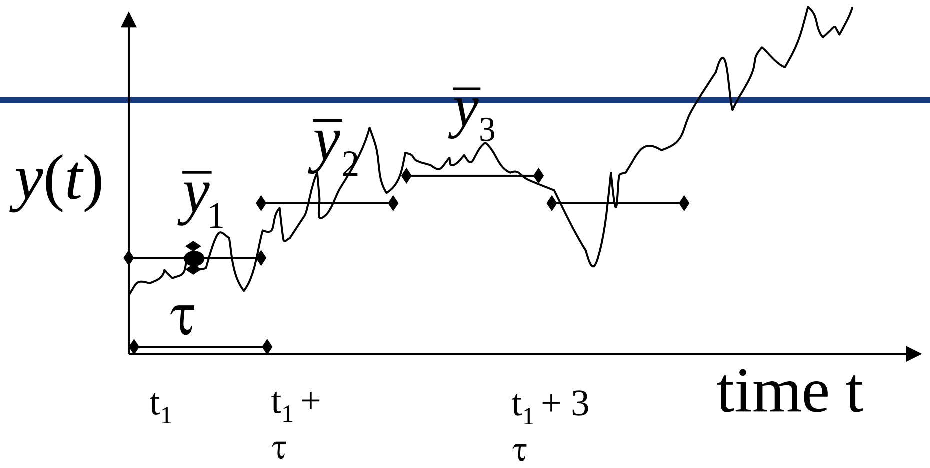
$$\sigma^2(\tau) \propto \frac{1}{\tau}$$

The variance depends on the observation interval τ

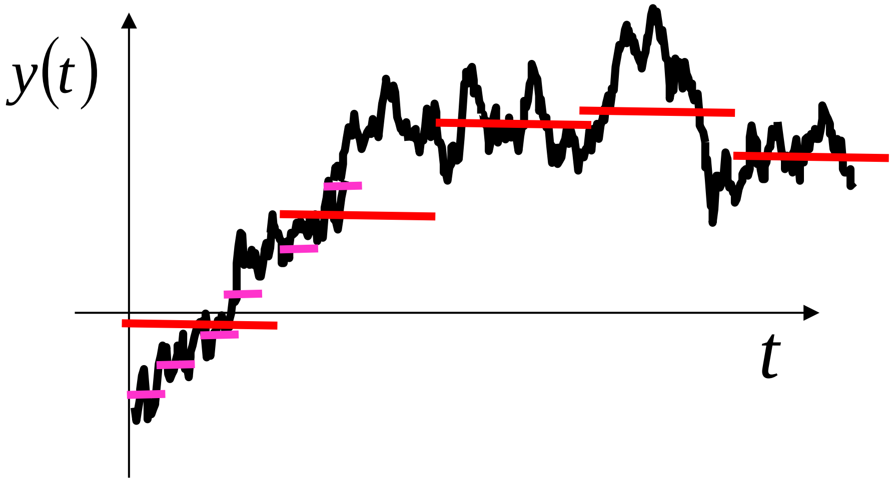
Instability - dispersion - variability

The variance is the measure of dispersion

$$\sigma^2(\tau) = \frac{\sum_{i=1}^{N} (\bar{y}_i - \mu)^2}{N}$$



Other typical noisy behaviour (long term noise)



If the "noise" is not strictly stationary, e.g. a random walk, the estimation of the variance do not converge and depends on N the number of samples (for any τ)

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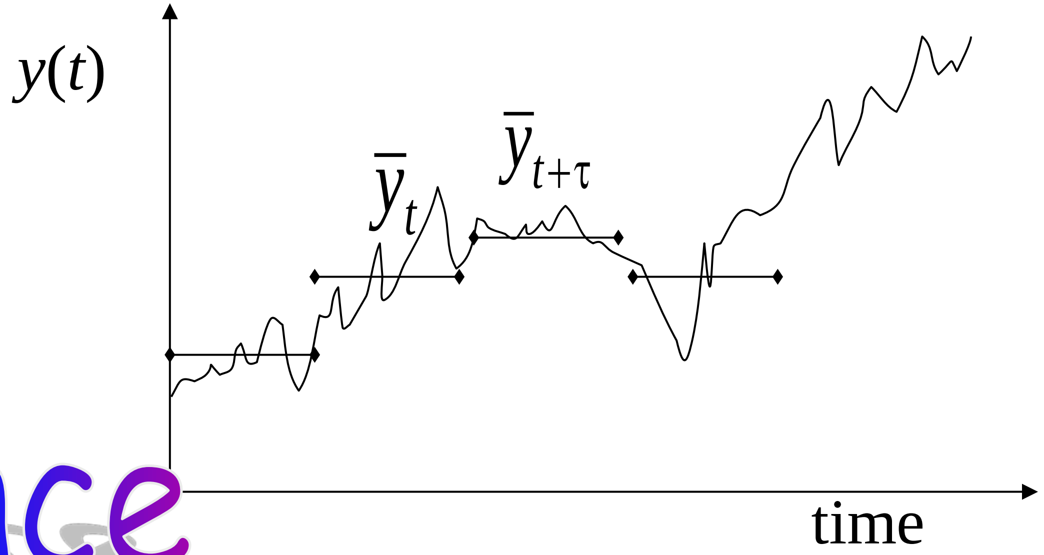
IDEA of Allan and Barnes (1966) let’s agree on the number of samples N=2

$$\sigma^2(N=2,\tau)=\frac{\sum_{i=1}^{2}(\bar{y}_i-\mu)^2}{2}=\frac{1}{2}\left(\bar{y}_1-\bar{y}_2\right)^2$$

and let’s average many 2-sample variances

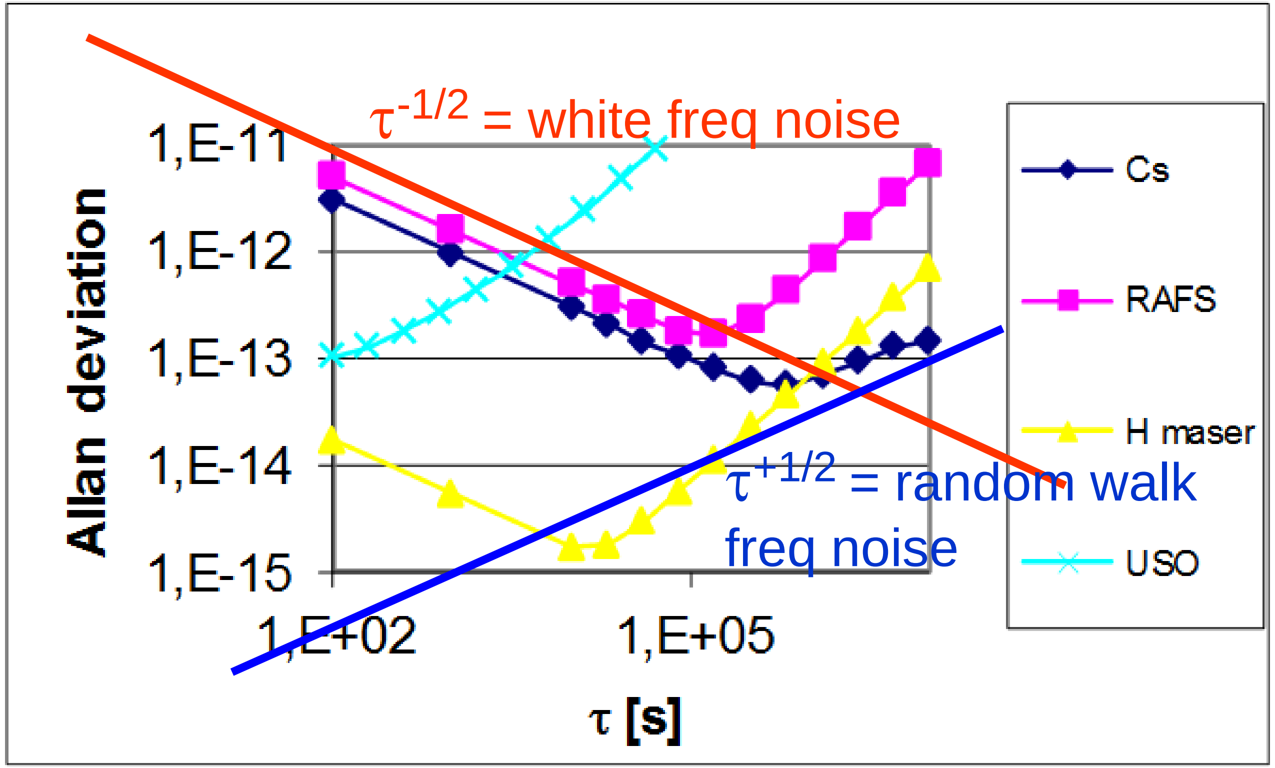
$$\sigma_y^2(\tau)=\frac{1}{2}\left\langle\left(\bar{y}_{t+\tau}-\bar{y}_t\right)^2\right\rangle$$



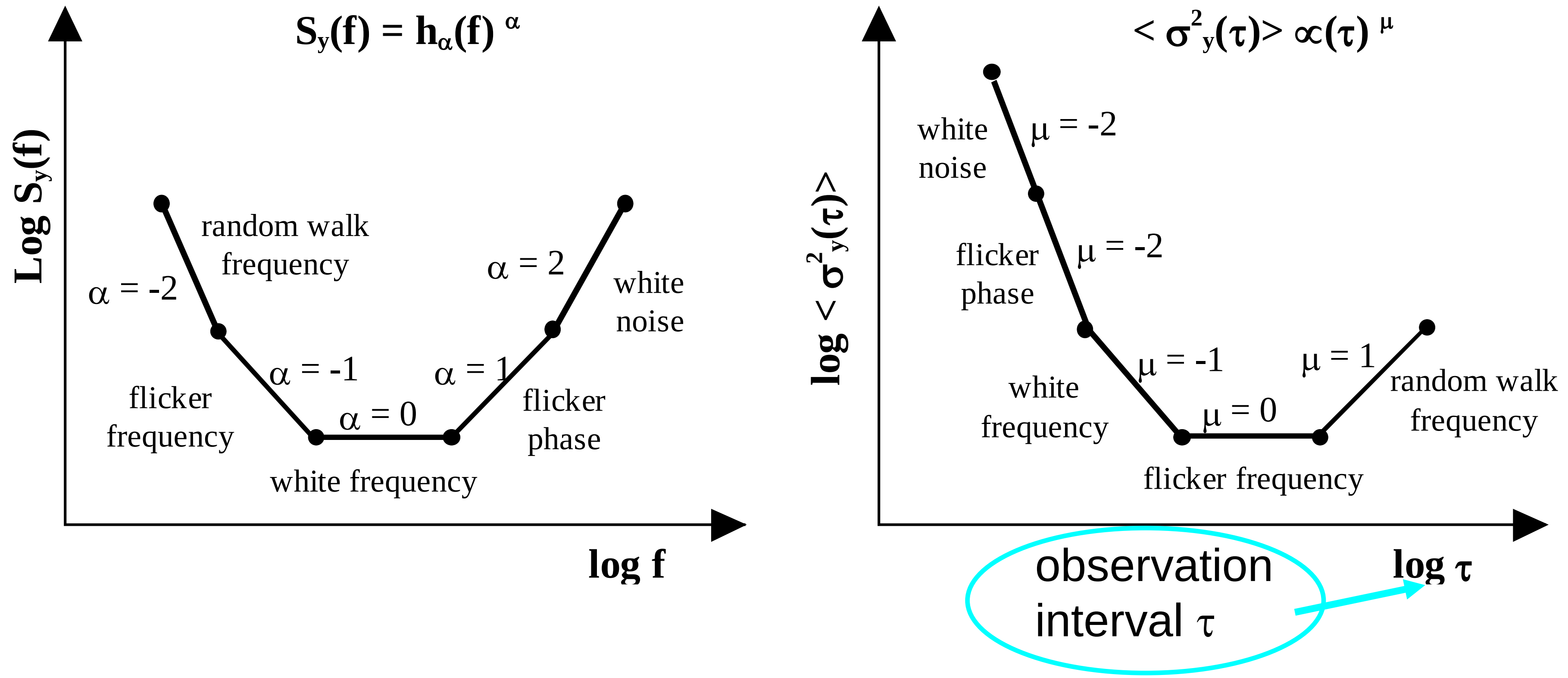
Allan variance

stability of the mean values (on τ intervals), actually variance of increments

With the Allan deviation we understand which type of noise is (mostly) affecting the measures, usually affecting the frequency of the atomic clocks



Allan variance <-> spectral density



What we learnt

1- In some cases, the frequency instability is more impacting than single measure uncertainty

2. The instability variance depends on the observation interval τ

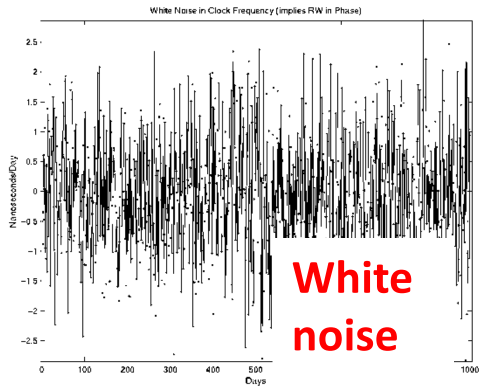
3. In some cases, the noise is non strictly stationary and the classical variance is not an appropriate tool. The Allan variance was proposed and important properties were found

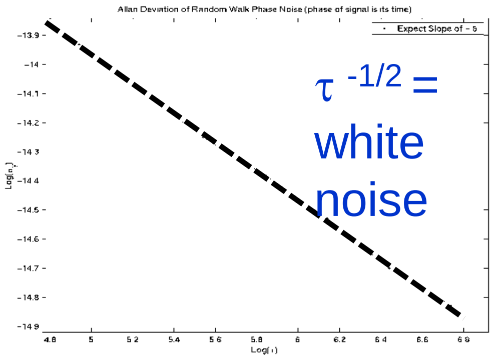
Most common noise are white, random walk, and integrated random walk

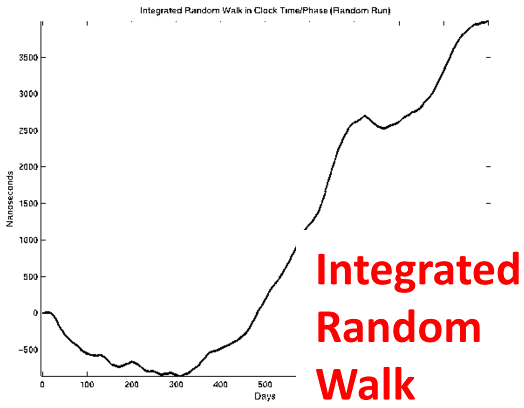
Time offset = Integral of

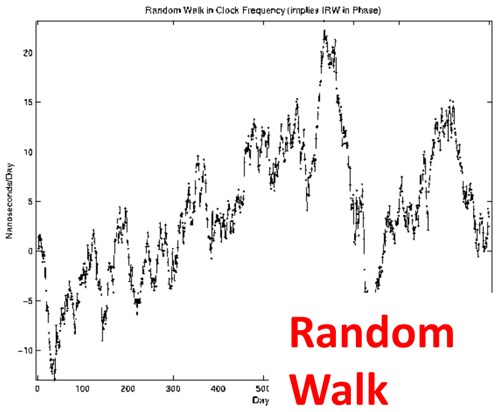
Frequency offset

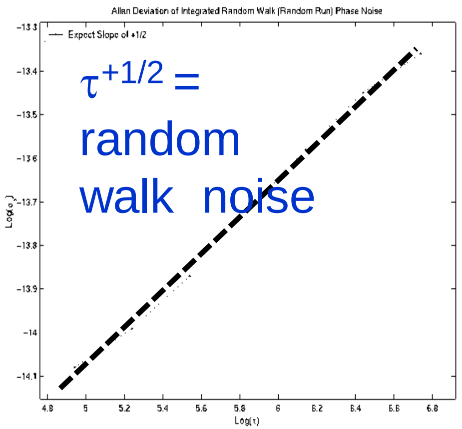
Allan Deviation











Often we want to estimate the accumulated time error due to stochastic components.? Is there a method?

Allan, UFFC 1987

IEEE TRANSACTIONS ON ULTRASONICS, FERROELECTRICS, AND FREQUENCY CONTROL, VOL. UFFC-34, NO. 6, NOVEMBER 1987

TABLE III

Typical Noise Types α	Name	Optimum Prediction $x(\tau_p)$ rms[a]	Time Error: Asymptotic Form
2	white-noise PM	$\tau_p \cdot \sigma_y(\tau_p)/\sqrt{3}$	constant
1	flicker-noise PM	$\sim\tau_p \cdot \sigma_y(\tau_p)\sqrt{\ln \tau_p / 2 \ln \tau_0}$	$\sqrt{\ln \tau_p}$
0	white-noise FM	$\tau_p \cdot \sigma_y(\tau_p)$	$\tau_p^{1/2}$
−1	flicker-noise FM	$\tau_p \cdot \sigma_y(\tau_p)/\sqrt{\ln 2}$	τ_p
−2	random-walk FM	$\tau_p \cdot \sigma_y(\tau_p)$	$\tau_p^{3/2}$

[a] τ_p is the prediction interval.

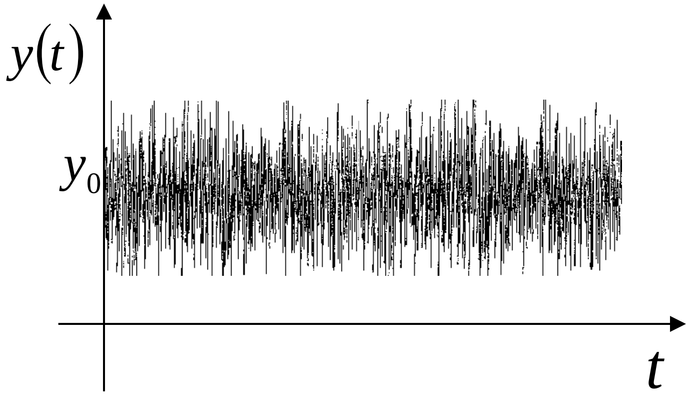
let's derive that

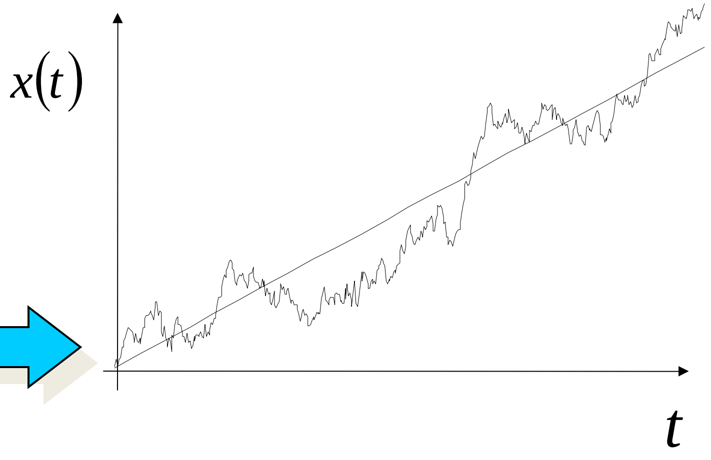
By the mathematical model for deterministic and stochastic behaviour

A mathematical model for determinist and stochastic behaviour

White noise plus constant offset and its integrated effect

Frequency offset / Velocity	Time offset / Position
$y(t) = y_0 + \xi(t)$	$x(t) = x_0 + y_0(t - t_0) + \int_{t_0}^{t} \xi(s)\,ds$





White Frequency Noise + constant initial offset

$$\xi(t) : N(0, \sigma^2)$$

Straight Line + Random Walk on Time Offset

Mathematical context: notations

Random Walk

Wiener Process

(Brownian Motion)

White Noise $\xi(t)$

"derivative" of a Wiener process

$$\xi(t) : N(0, \sigma^2)$$

$$\xi(t)dt = \sigma \, dW(t)$$

$$x(t) = x_0 + y_0(t - t_0) + \int_{t_0}^{t} \xi(s)\,ds$$

$$y(t) = \frac{dx(t)}{dt} = y_0 + \xi(t)$$

White noise is the derivative of Wiener process

$$\xi(t)dt = \sigma\, dW(t)$$

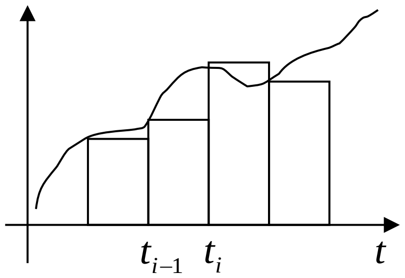
Stochastic calculus

$$dx(t) = y_0 dt + \sigma\, dW(t)$$

Riemann integral

Stochastic integral

$$x(t) = \underbrace{\int_{t_0}^{t} y_0 dt}_{\text{Riemann integral}} + \sigma \underbrace{\int_{t_0}^{t} dW(t)}_{\text{Stochastic integral}}$$



Stochastic integral

$$\approx \lim_{\delta \to 0} \sum_{i=1}^{n} \left(W(t_i) - W(t_{i-1})\right)$$

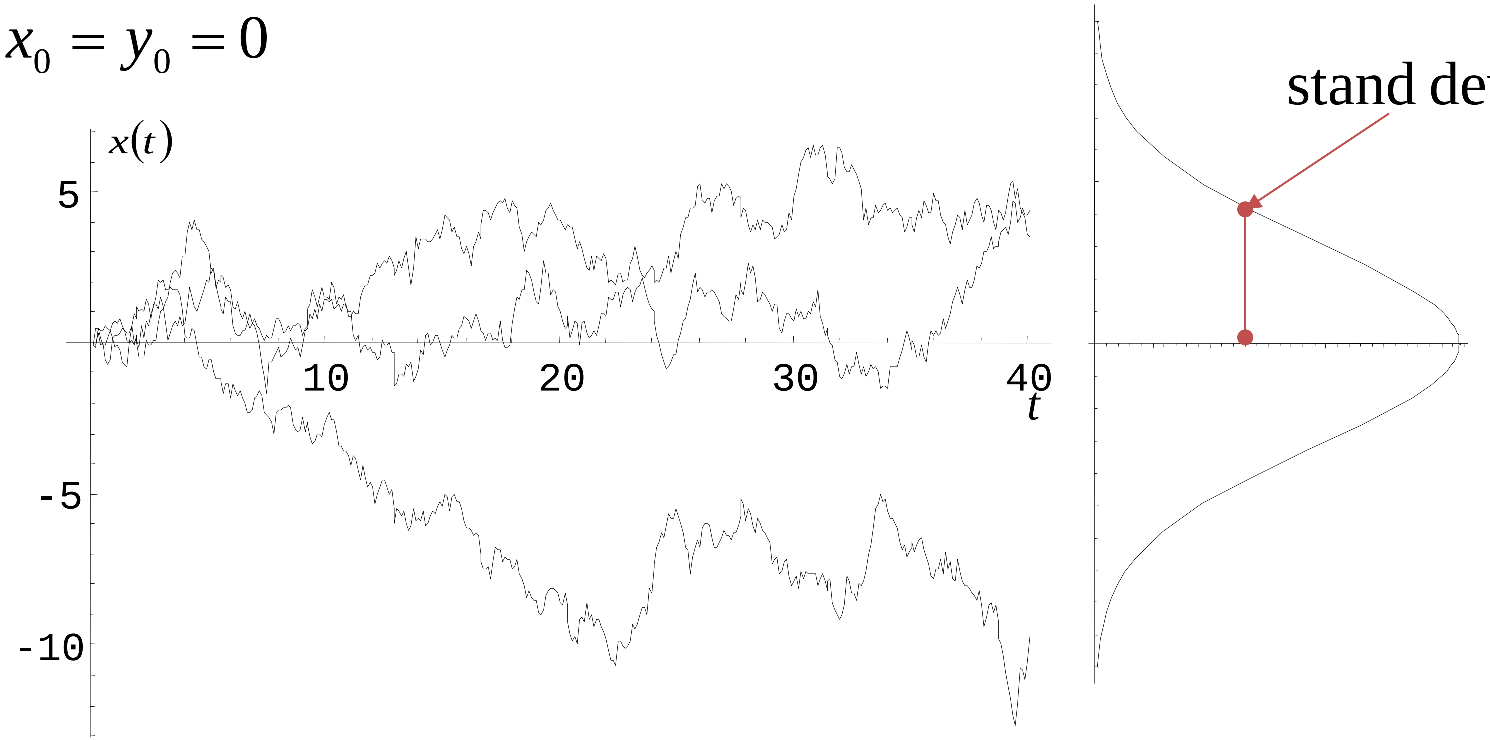
$$\delta = t_i - t_{i-1}$$

It depends on the choice of t_i ⇒ **Itô stochastic integration**

Random Walk position x(t)= σ W(t)

Diffusion coefficient
linked to Allan Deviation

$$x_0 = y_0 = 0$$



$$\text{stand dev} = \sqrt{\sigma^2 \cdot t}$$

$$W(t) = \int_0^t \xi(s) ds$$

The Wiener process is the integration (accumulation) of random steps

The probability of position at epoch *t* can be evaluated

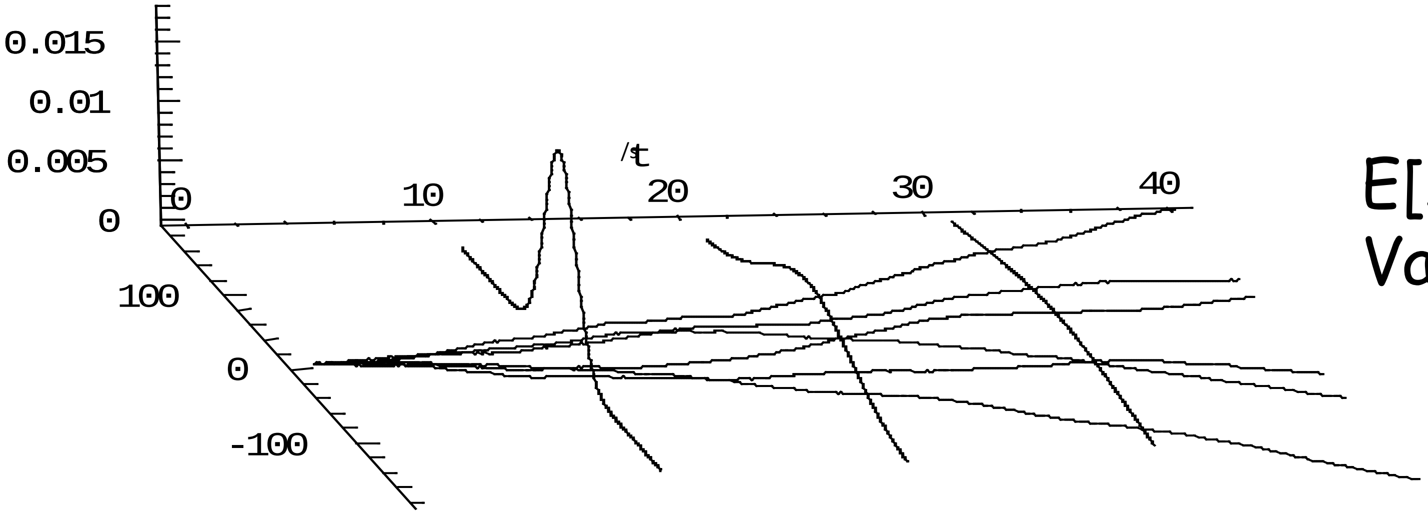
G. Panfilo, P. Tavella, "Atomic Clock Prediction based on stochastic differential equations", Metrologia 45, 6, (2008) S108-S116

P. Tavella, C. Zucca “The Clock Model and its Relationship with the Allan and related Variances”, IEEE Trans. UFFC Ultras. Ferroel. Freq. Control, vol. 52, no. 2, Feb. 2005, pp. 288-295

D. Allan, “Time and frequency (time-domain) characterization, estimation, and prediction of precision clocks and oscillators”, 1987;34(6):647-54. doi: 10.1109/t-uffc.1987.26997.

FOR ANY NOISE

At time t after synchronisation, the time error x(t) is described by a Gaussian probability density with



$E[x(t)]=0$

$\text{Variance}[x(t)]=\sigma^2 \cdot t$

Diffusion coefficient linked to Allan Deviation
$\sigma^2\, t= \text{AVAR}(t) \cdot t^2$

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[a] τ_p is the prediction interval.

We do not know the exact time error of an individual clock but we can estimate its statistical overall behaviour

In case of more noises: integrated white and random walk the exact solution exists:

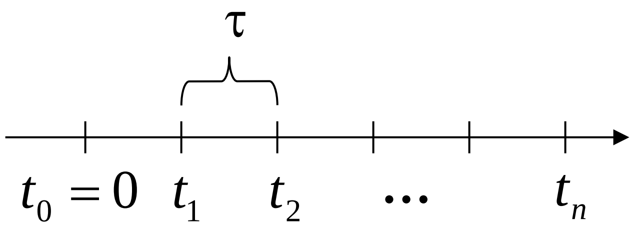
$$\begin{cases} x_1(t) = x_0 + y_0 t + a\dfrac{t^2}{2} + \sigma_1 W_1(t) + \sigma_2 \int_0^t W_2(s)ds \\ x_2(t) = y_0 + a\,t + \sigma_2 W_2(t) \end{cases}$$

White Freq. noise

RW Freq. noise

Iterative solution useful

for **simulations**, filter, …



$$\begin{cases} X_1(t_{k+1}) = X_1(t_k) + X_2(t_k)\tau + a\dfrac{\tau^2}{2} + \sigma_1 \mathrm{W}_{1,k}(\tau) + \sigma_2 \boxed{\int_{t_k}^{t_{k+1}} W_2(s)ds} \\ X_2(t_{k+1}) = X_2(t_k) + a\,\tau + \sigma_2 \boxed{\mathrm{W}_{2,k}(\tau)} \end{cases}$$

correlated processes

with initial conditions

$$\begin{cases} X_1(0) = x_0 \\ X_2(0) = y_0 \end{cases}$$

L. Galleani, L. Sacerdote, P. Tavella, C. Zucca, "A mathematical model for the atomic clock error", Metrologia Volume 40 (3), 2003, S257-S264

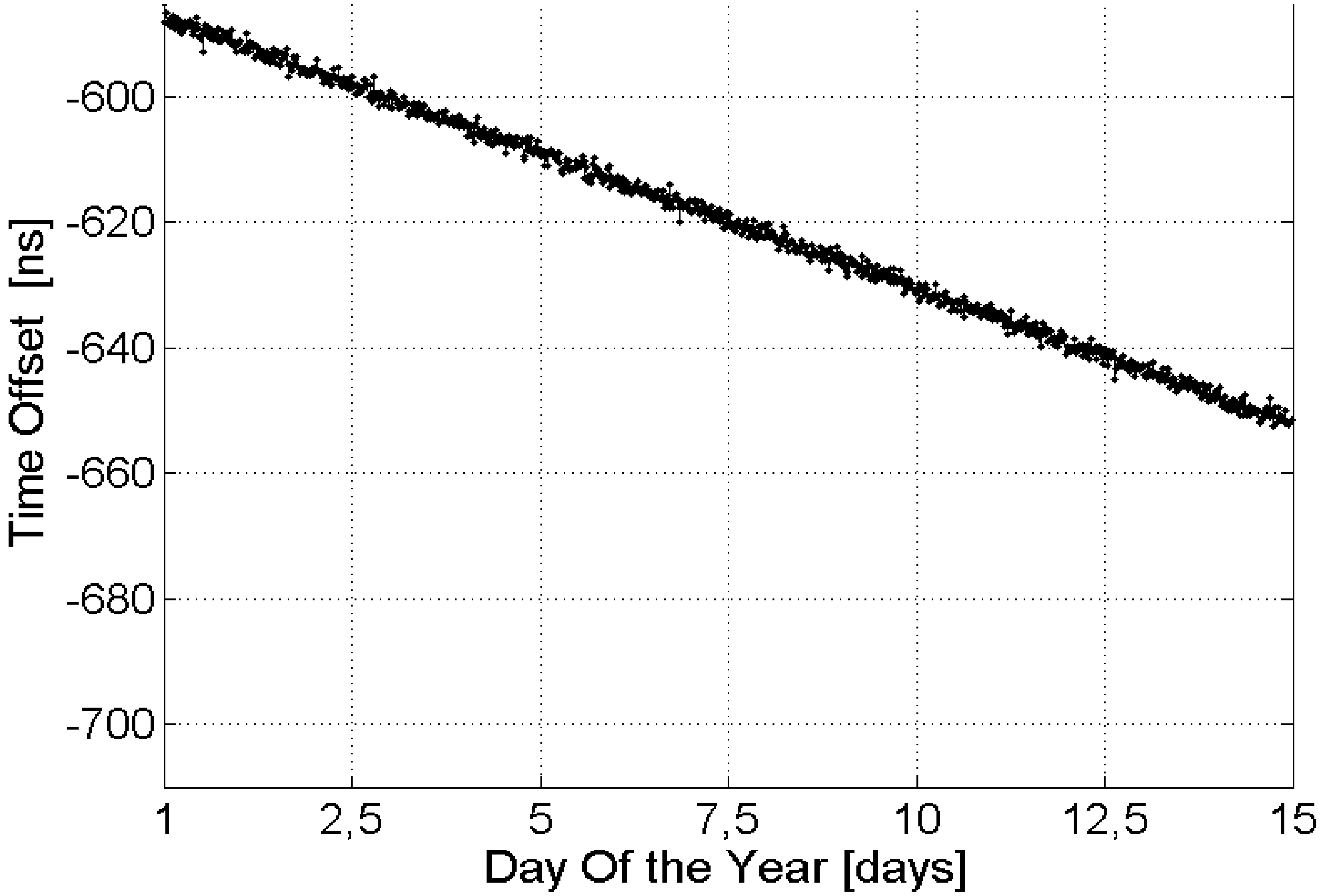
What we learnt

1- Deterministic and stochastic behaviour can be modeled by stochastic differential equations(SDE)

2. The exact (or approximated) solution of the SDE allows the dynamic behaviour estimation, simulation, and prediction

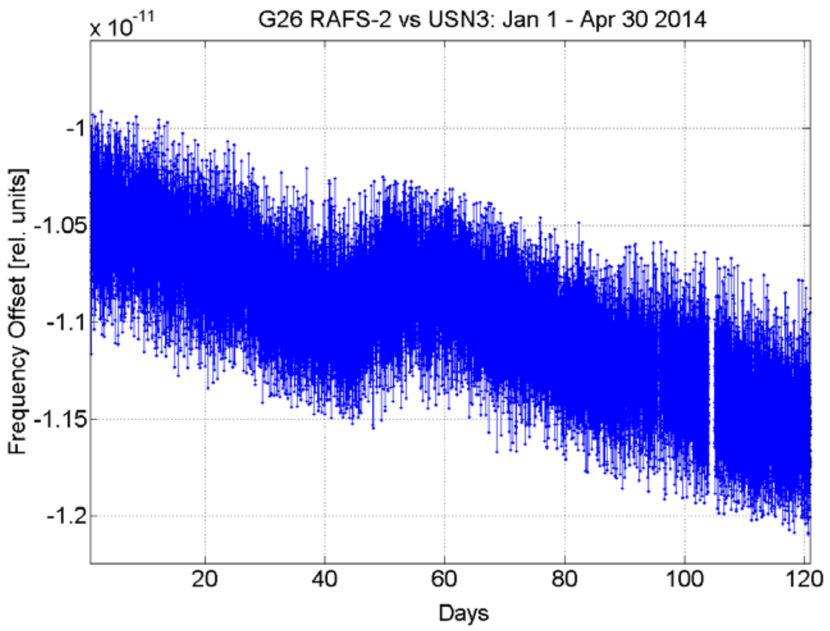
3. The noises imbedded in the model has usually zero mean value, they do not impact on the prediction but on the uncertainty of the predicted values
(therefore allowing to evaluate confidence intervals)

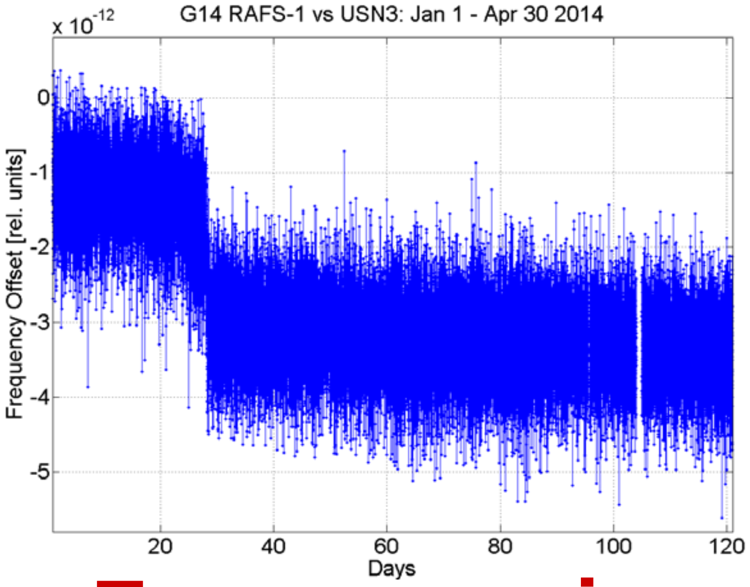
Will the well understood and modeled behaviour last forever?

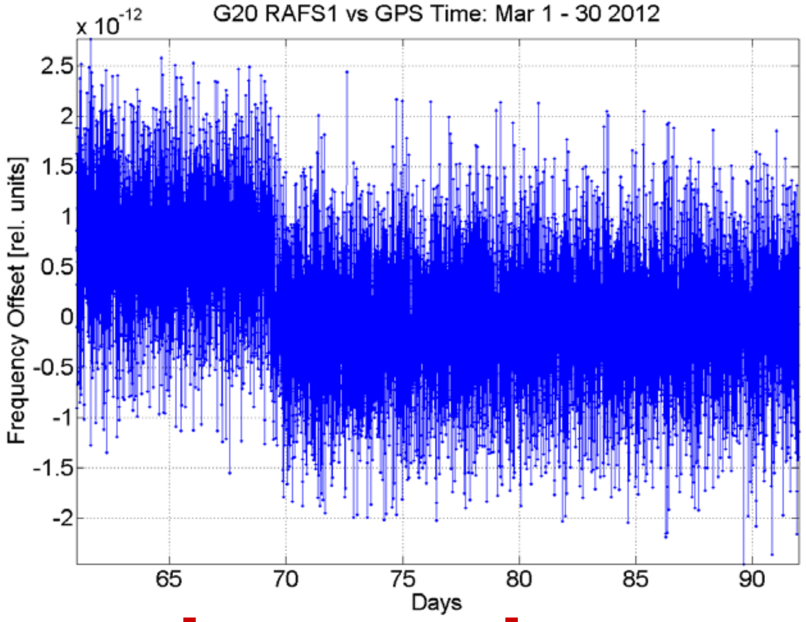


Example of GPS space clocks

Possible causes of nonstationarities on space clocks are manoeuvres and tests on board, environmental variations, eclipses, etc. In other cases, the nonstationarities may be due to the clock itself.

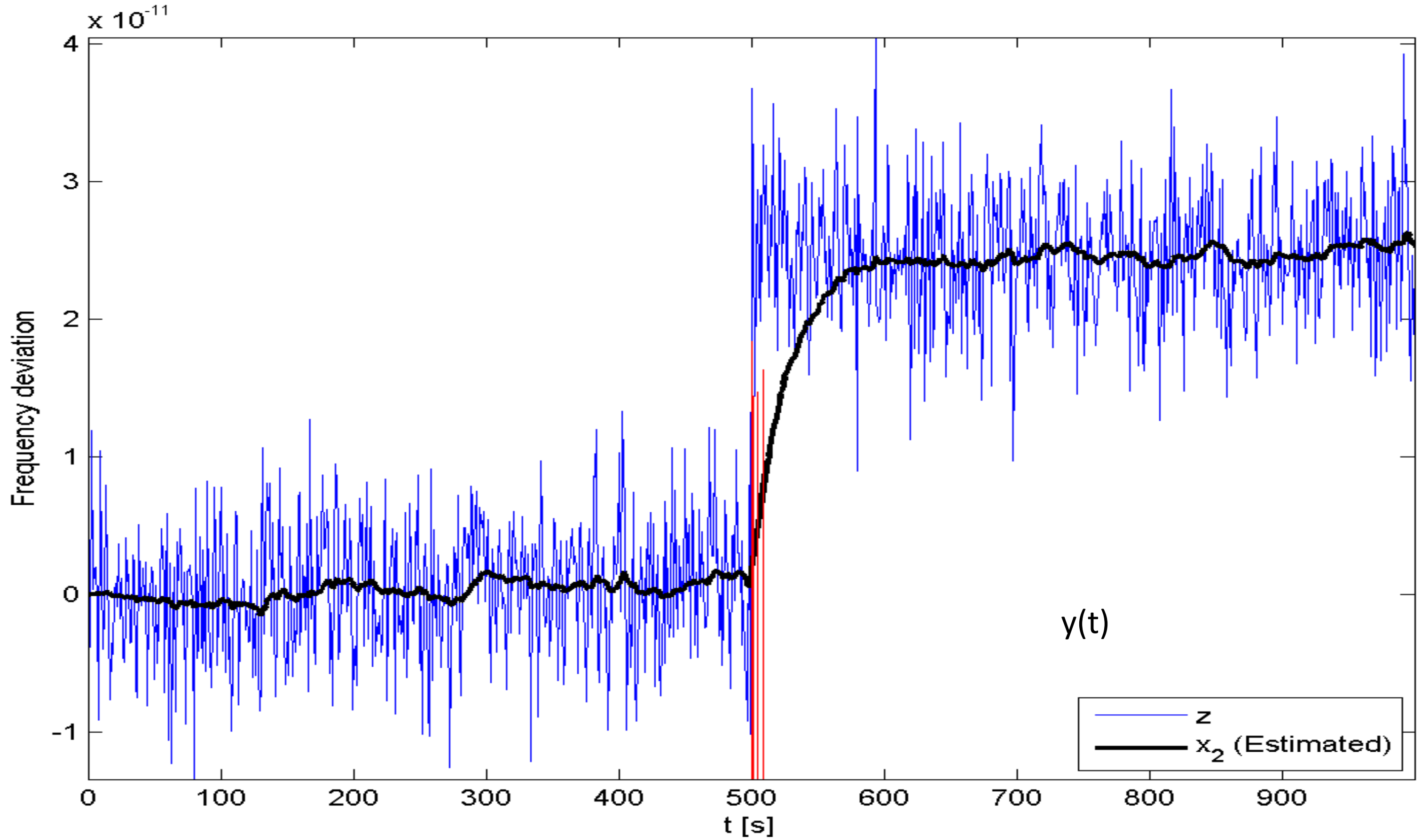






Frequency jumps are observed

WE NEED A FREQUENCY JUMP DETECTOR



Q. Wang, F.Droz, P. Rochat, "Robust Clock Ensemble for Time and Frequency Reference System", presented at the EFTF/IFCS Denver 2015

L.Galleani,P.Tavella."Detection of Atomic Clock Frequency Jumps with the Kalman Filter," IEEE Transactions on Ultrasonics, Ferroelectr, Freq Control March 2012. vol. 59, no. 3, p. 504-509, March 2012

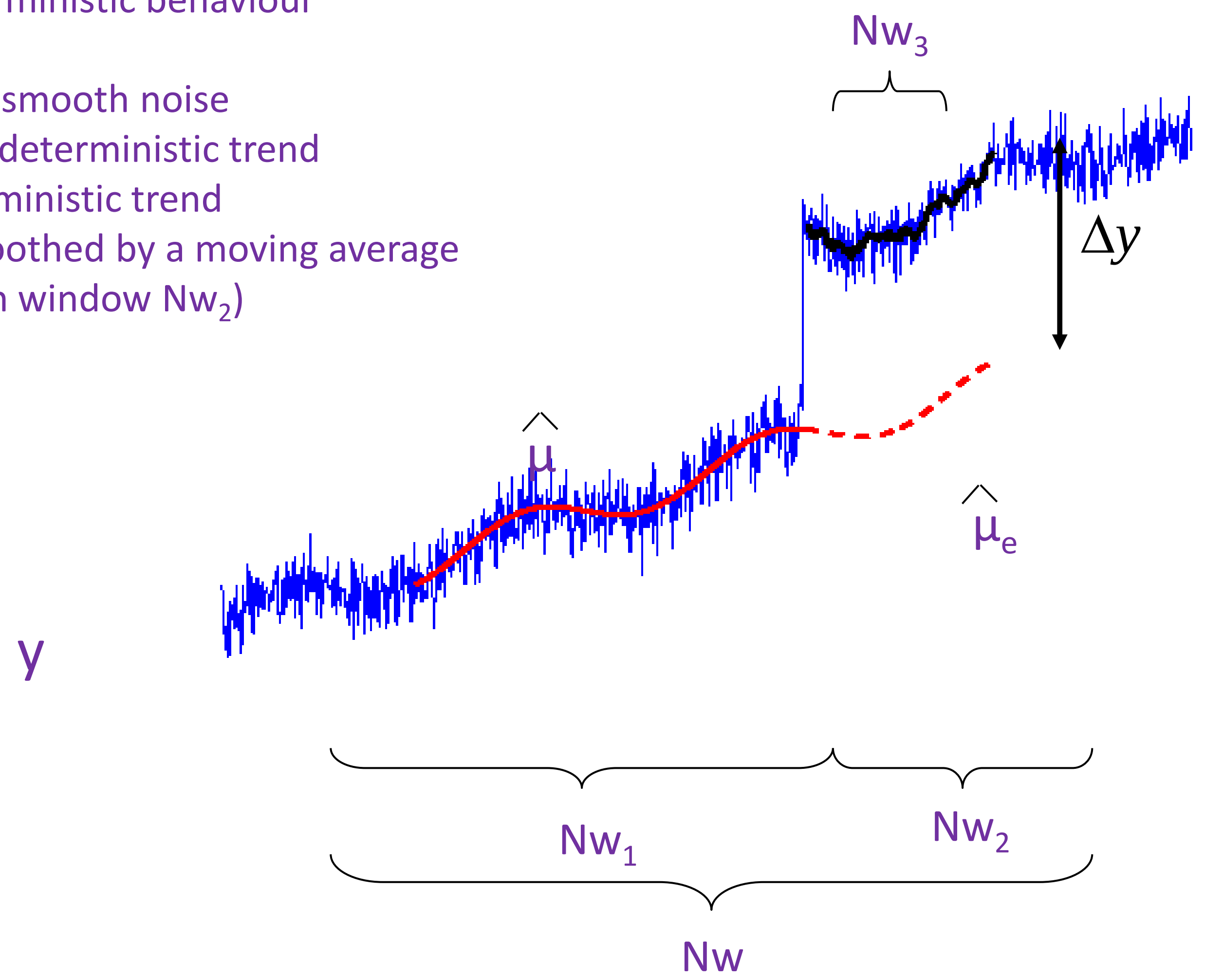
Huang X, Gong H, Ou G, Detection of weak frequency jumps for GNSS onboard clocks, IEEE Trans Ultrasonics, Ferroelectr Freq Control. 2014 May;61(5):747-55

From the past we best predict the future and then we compare prediction with fresh measures (or mean of measures)

Nw = full data set
Nw_1 = estimation of deterministic behaviour
Nw_2 = check on jump
Nw_3 = moving average to smooth noise
$\hat{\mu}$ = estimated value of deterministic trend
$\hat{\mu}_e$ = extrapolated deterministic trend
y_s = frequency values smoothed by a moving average on Nw_3 samples (sliding in window Nw_2)
$\Delta y = | y_s - \hat{\mu}_e |$

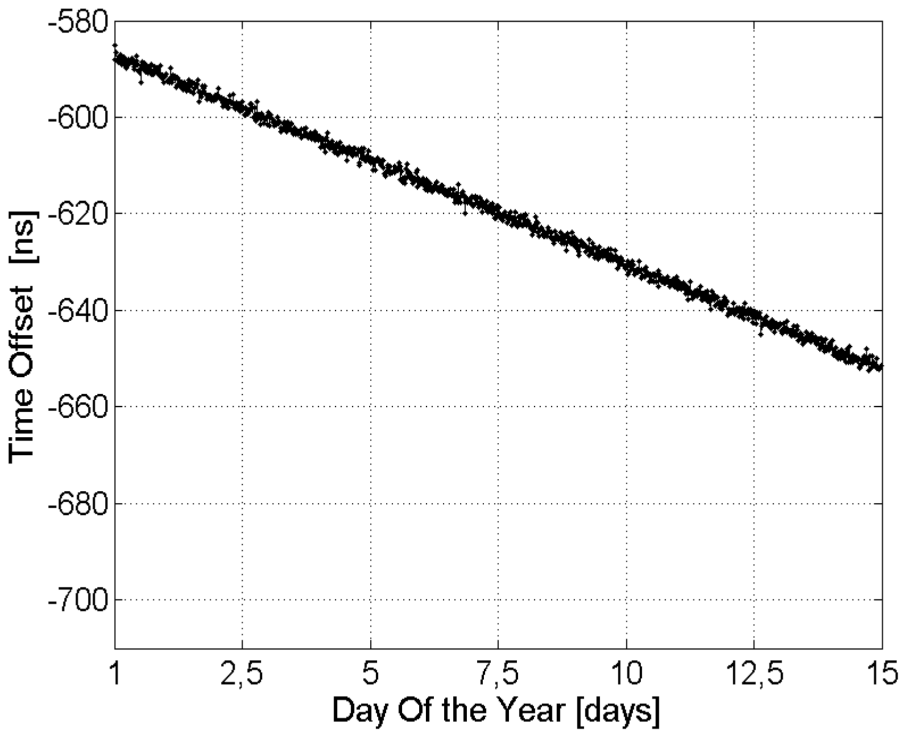
If Δy > threshold

→ Alarm

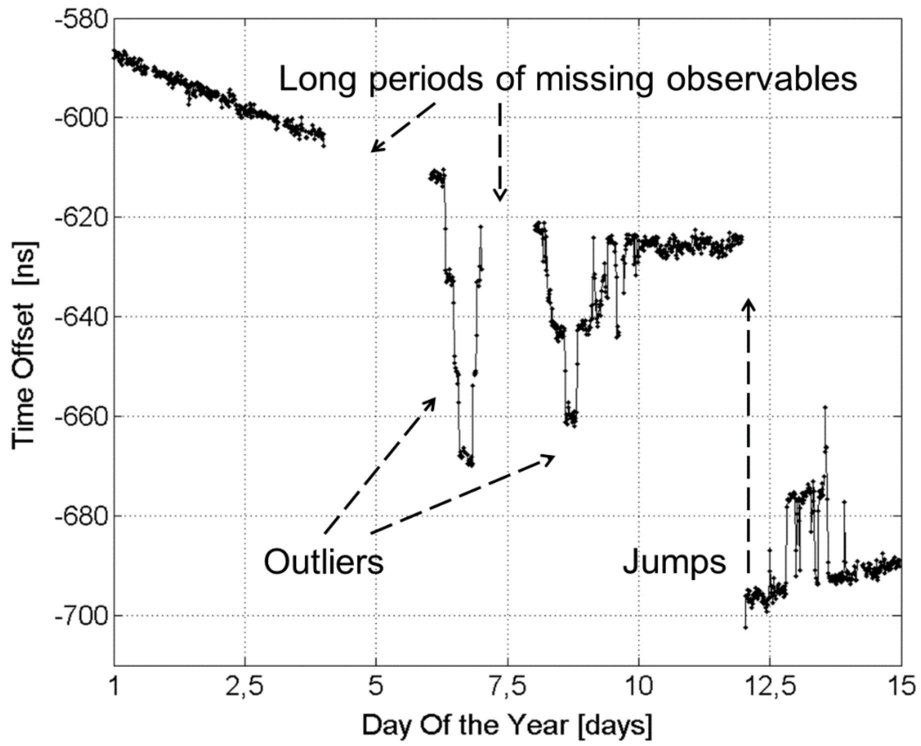


Sometimes real experimental data are not as we would like

ideal case



real case

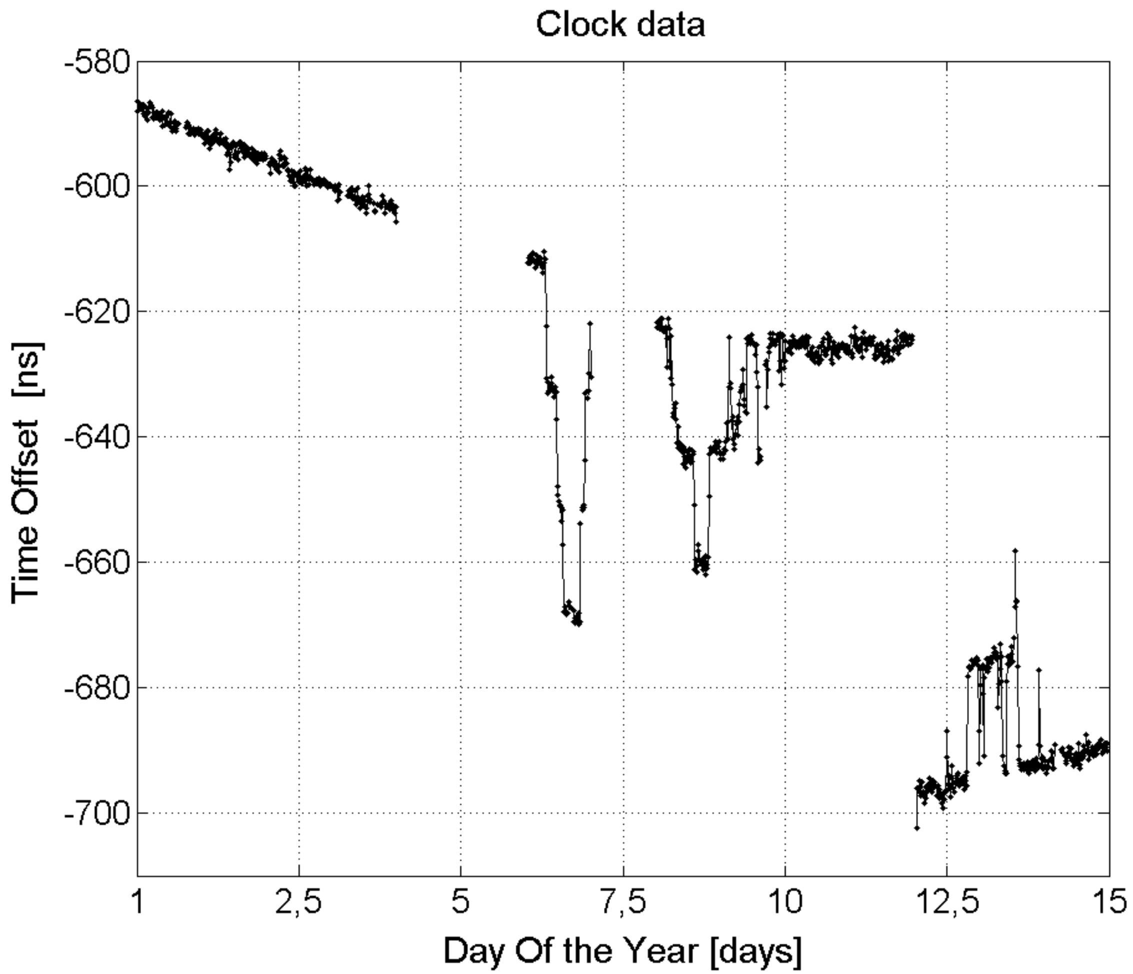


The importance of data preprocessing
(in automated analysis)

- ✓ Data series may present gaps, long periods of missing observation, outliers and jumps
- ✓ Anomalies have to be detected and properly removed before the data processing and analysis steps
- ✓ An efficient preprocessing is fundamental to ensure a proper data analysis

Outliers filtering

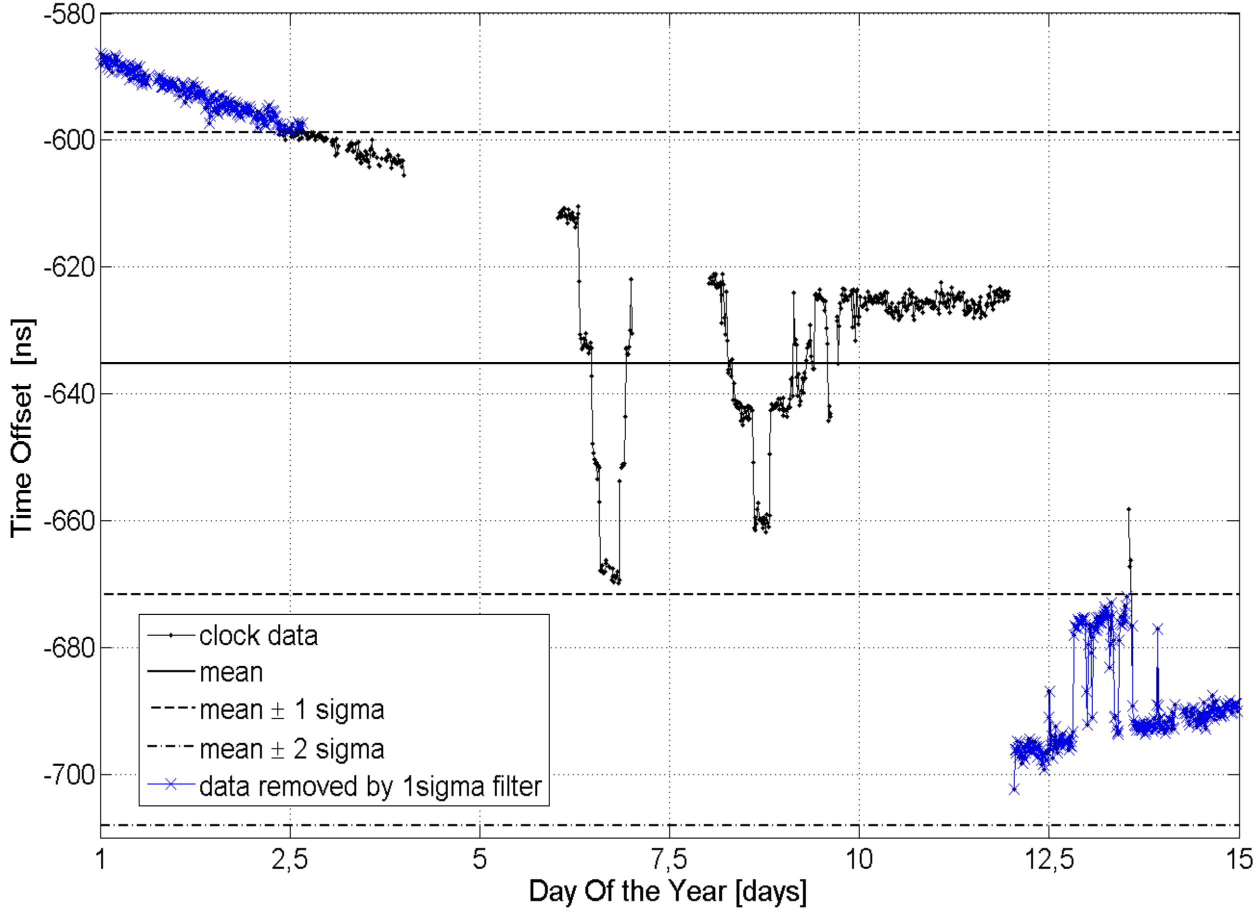
Examples of applications to experimental clock data



Outliers filtering

Examples of applications: batch sigma filter

Data outside (mean ± k σ) are removed



Outliers filtering

Examples of applications: batch MAD-based filter
Data outside (median ± k MAD) are removed

Median

Middle value separating the greater and lesser halves of a data set

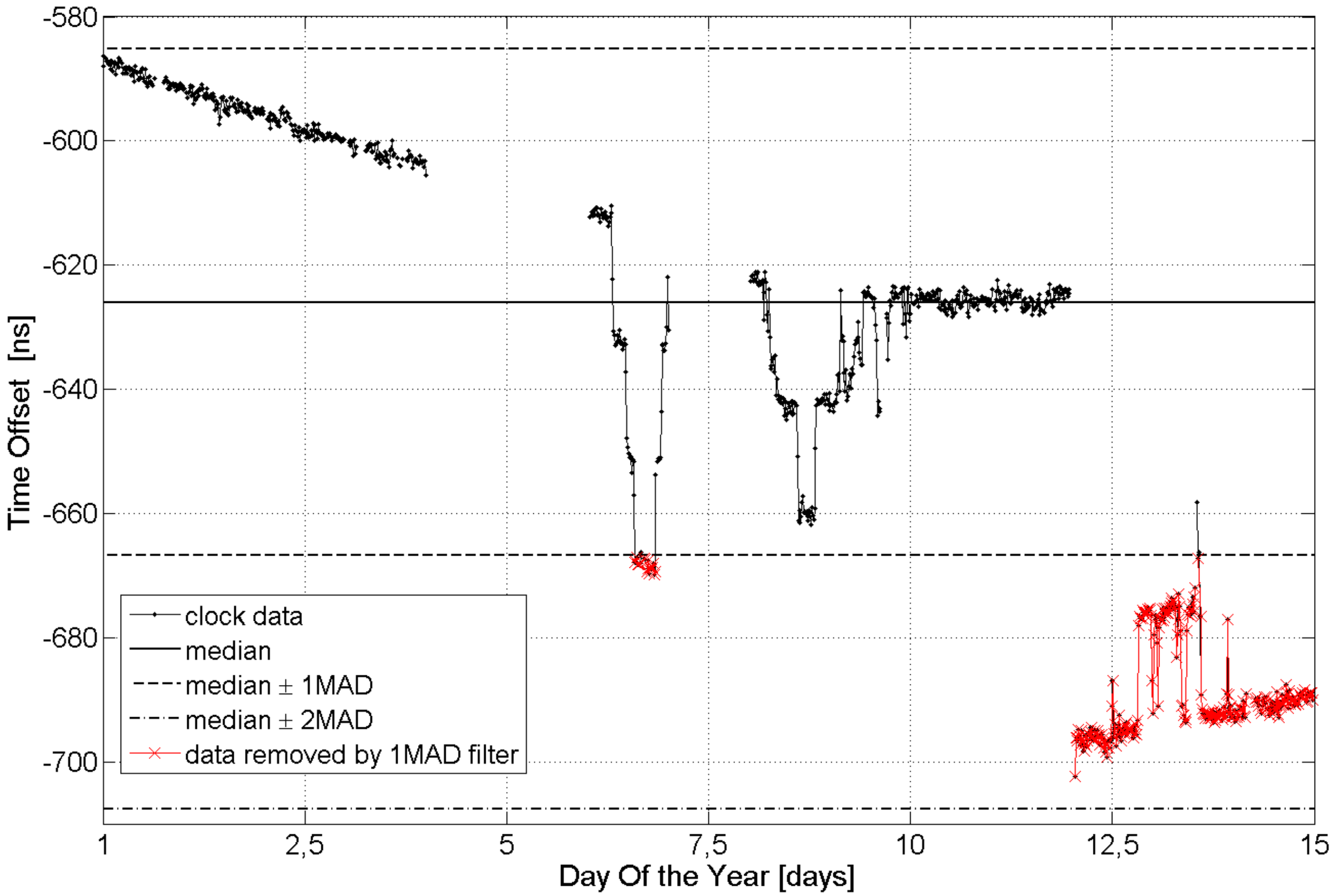
Ex: 2, 4, 7, 2, 9, 3, 1

1, 2, 2, **3**, 4, 7, 9

MAD

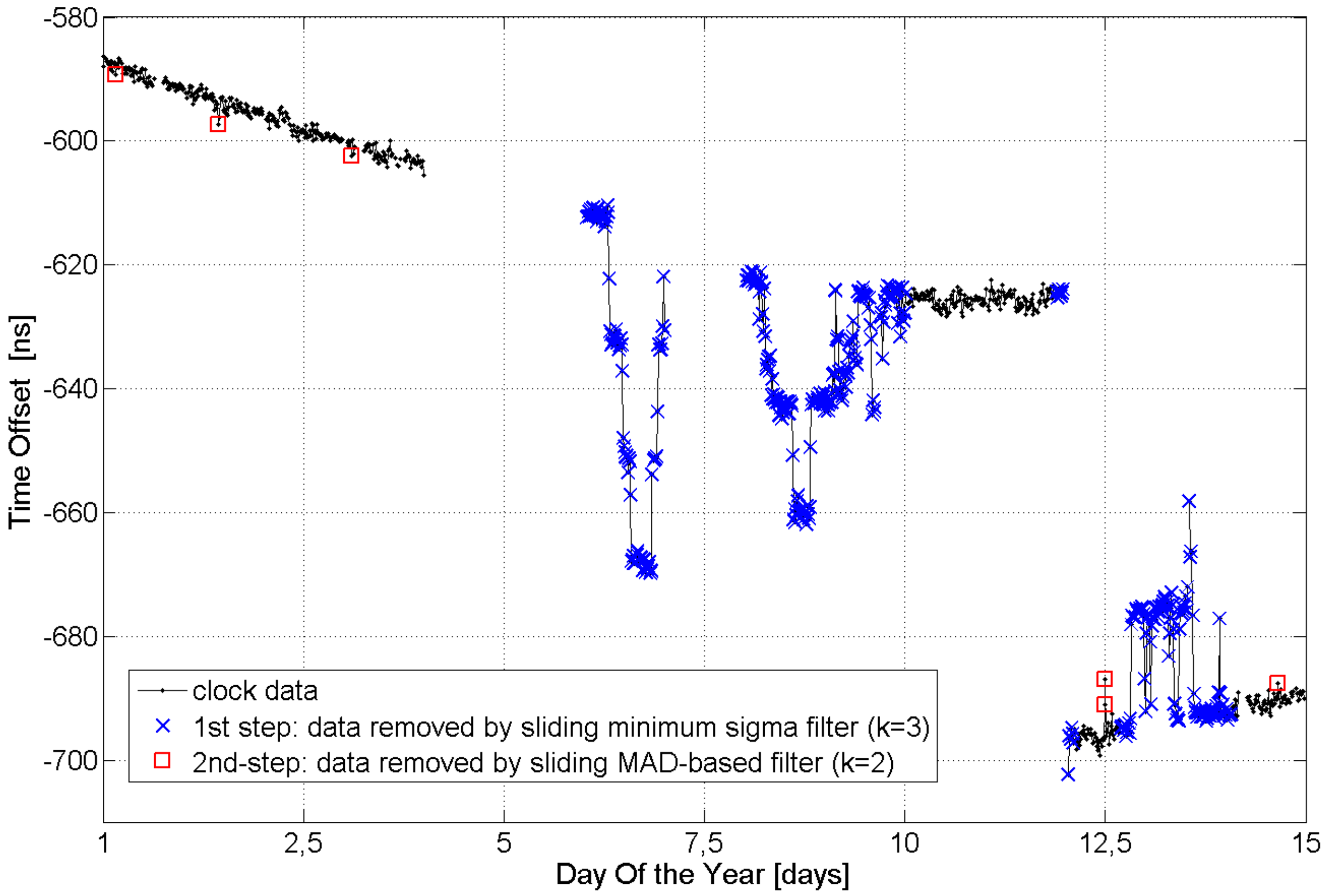
median of the absolute deviations from the data's median

MAD = Median ($|X_i$ - Median$(X_i)|$)



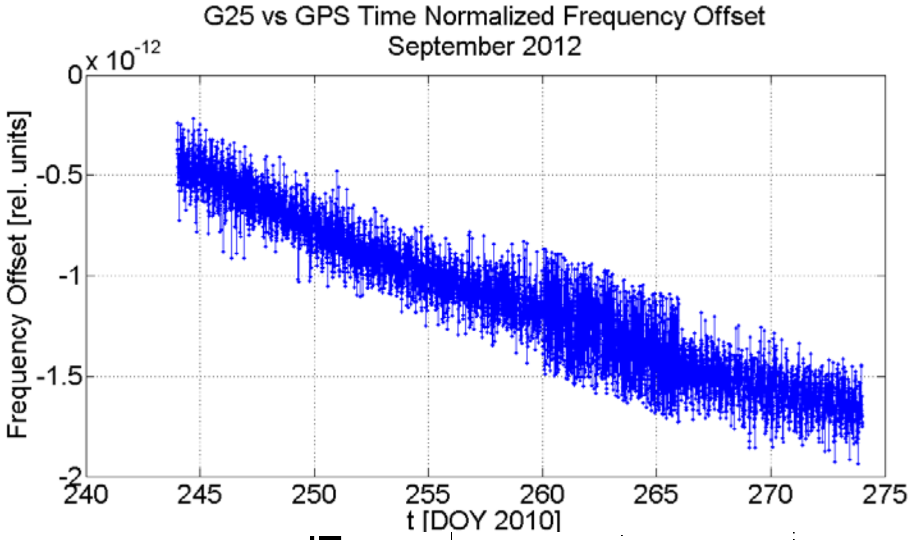
Outliers filtering

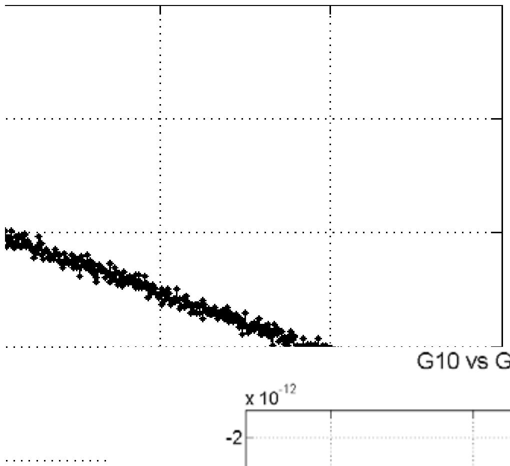
A moving window re evaluating sigma and MAD and validating outliers on different windows may be useful

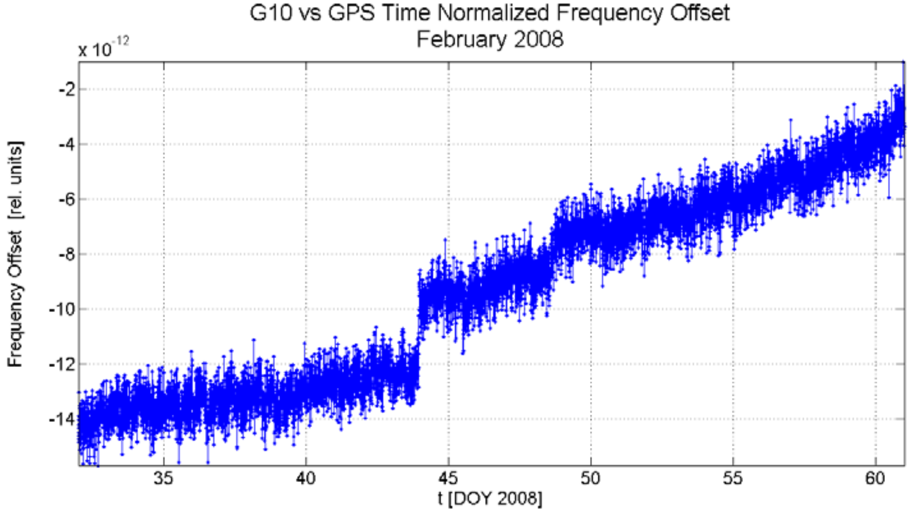


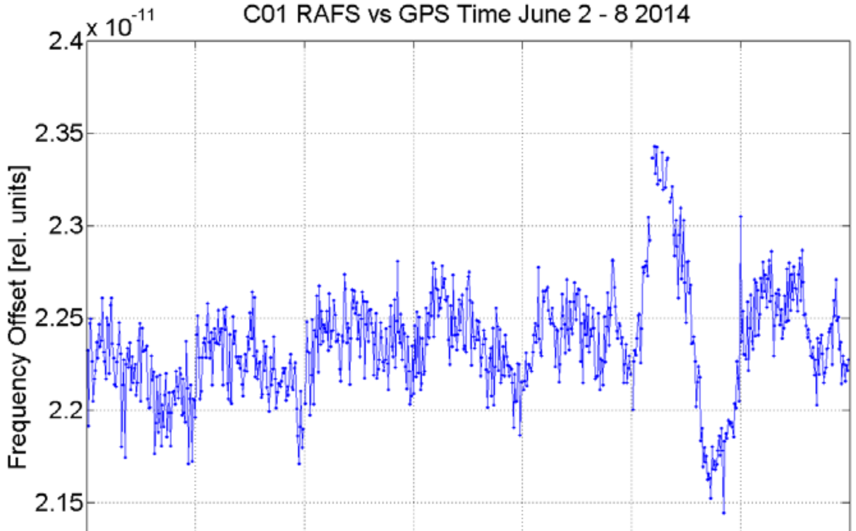
Filters are applied on a sliding window of 5 hours. Outliers are validated before removal.

Will the well understood and modeled behaviour last forever?









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We need a

dynamical characterization of the noise

Time and Frequency spectral analysis

for example

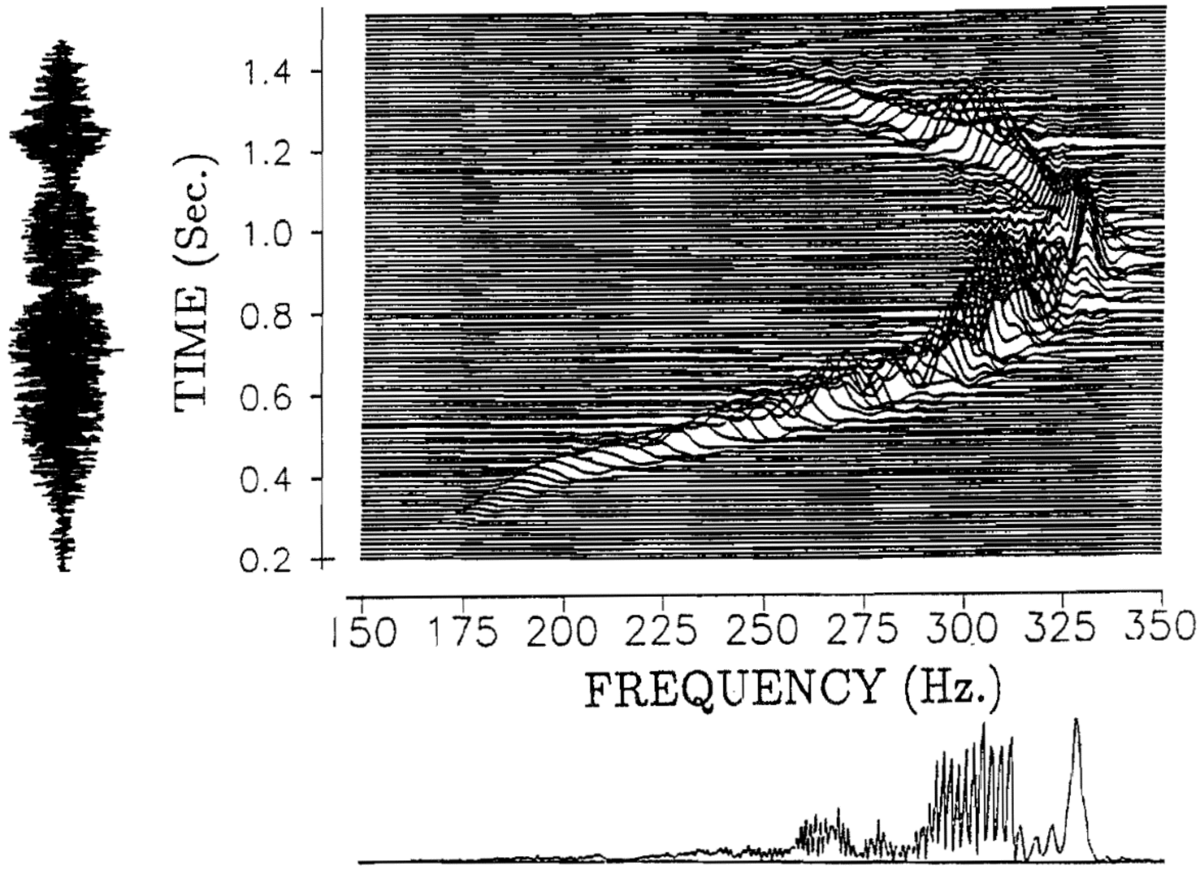
Not only estimating which frequencies existed

But also estimating when they existed

Time-frequency analysis

It describes how the frequencies of a signal change with time

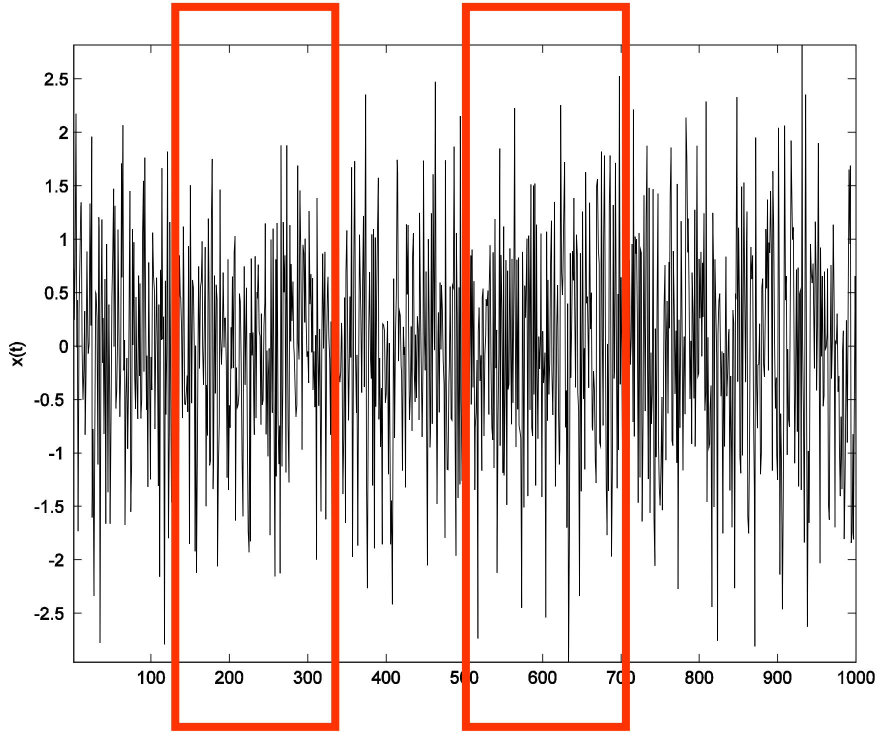
Bowhead whale



L. Cohen, *Time-frequency analysis*, Prentice-Hall, 1995

A Dynamic Allan variance

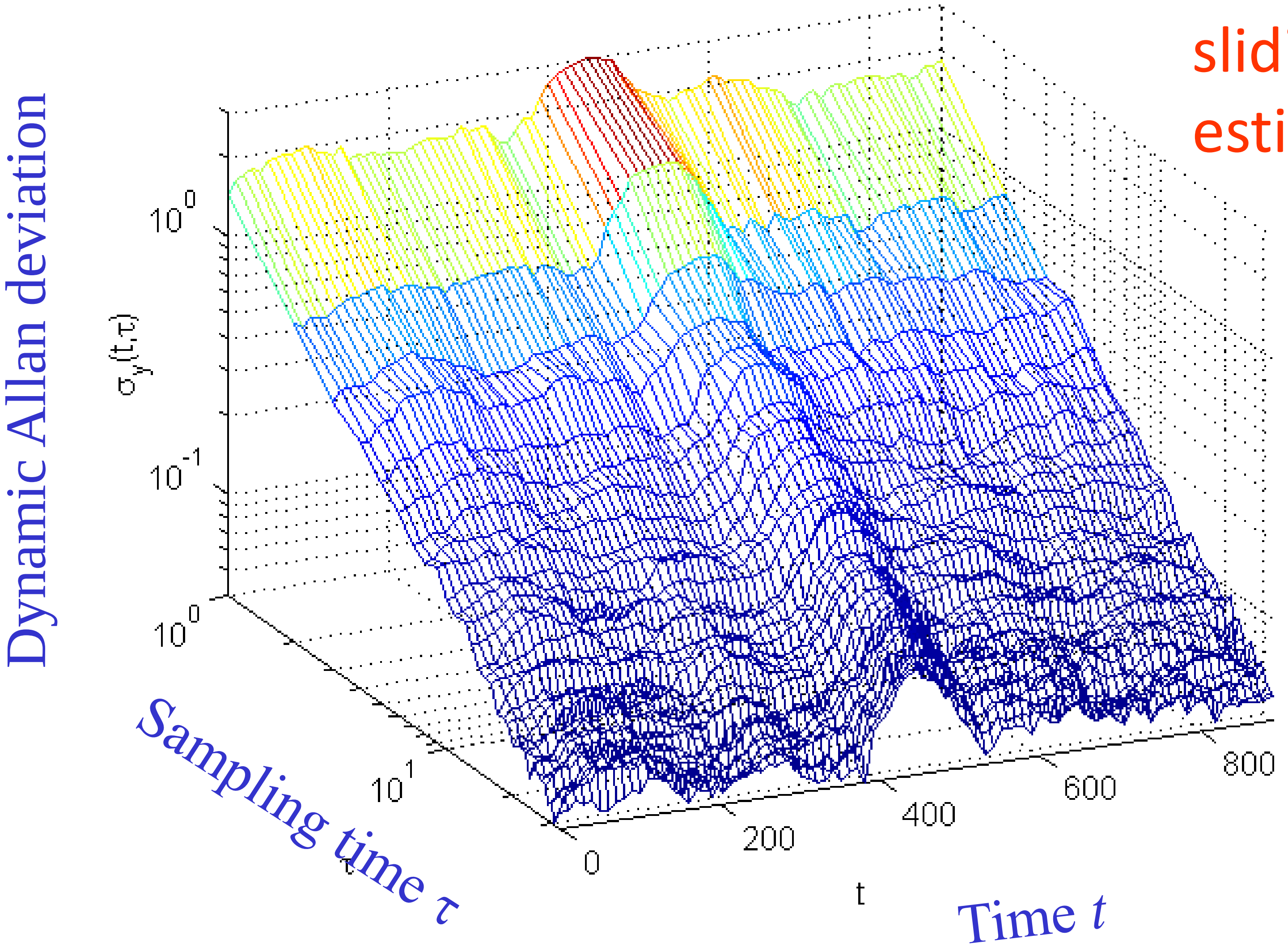
sliding the Allan variance estimator on the data



t_1 t_2

$\sigma_y(t_1,\tau)$ $\sigma_y(t_2,\tau)$ ... $\Rightarrow$ $\sigma_y(t,\tau)$

stability may vary with time

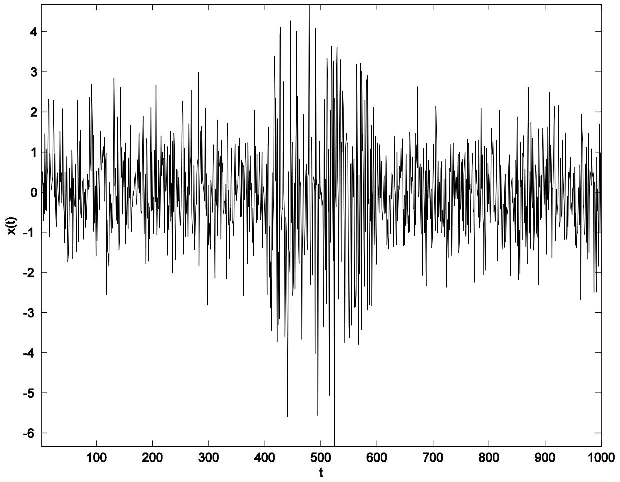


sliding stability estimator

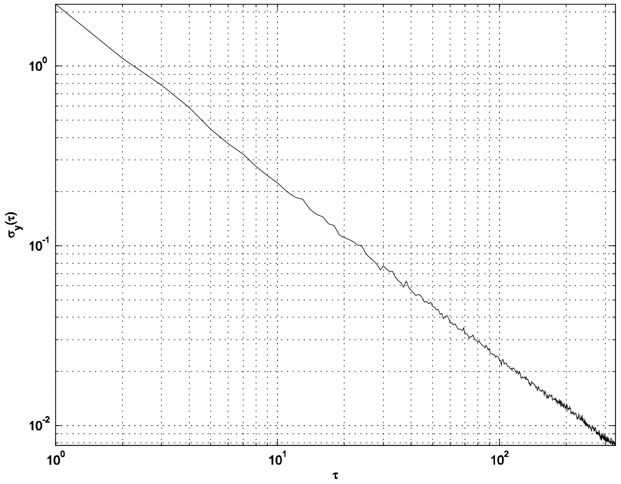
41

L. Galleani, P. Tavella, "Dynamic Allan variance" , IEEE Trans UFFC, vol. 56, no. 3, March 2009, pp450-464

Simulation results : Bump

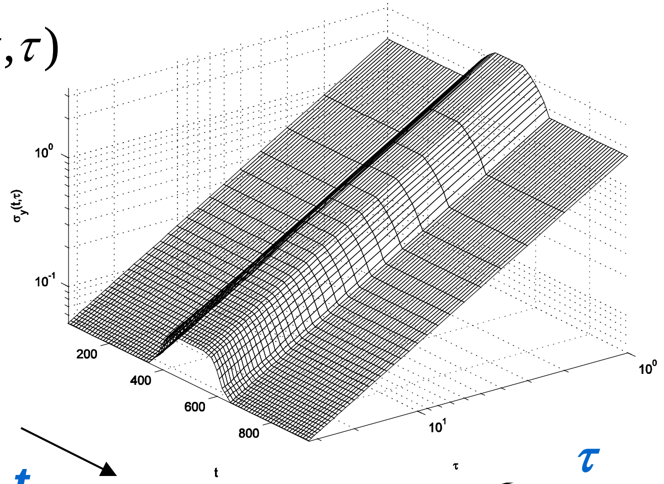


t



τ

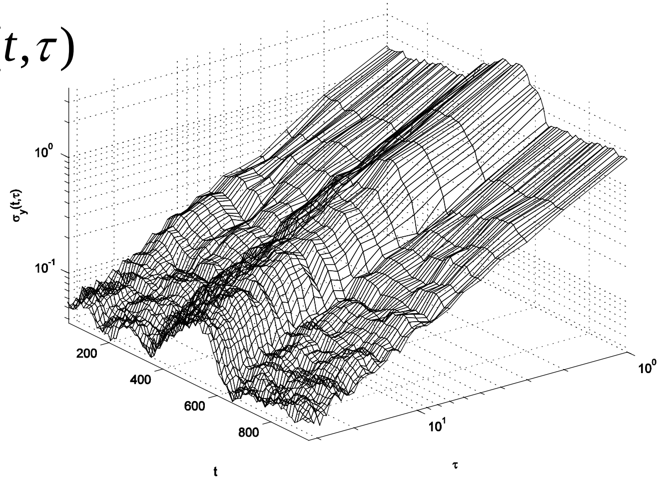
$\sigma_y(t,\tau)$



t

τ

$\hat{\sigma}_y(t,\tau)$



The Dynamic Allan variance

Discrete time formulation from the phase samples $x[n]$

$$\sigma^2{}_y[n,k] = \frac{1}{2k^2\tau_0{}^2}\frac{1}{Nw-2k}\sum_{m=n-Nw/2+k}^{n+Nw/2-k-1} E\left[\left(x_N[m+k]-2x_N[m]+x_N[m-k]\right)^2\right]$$

where:

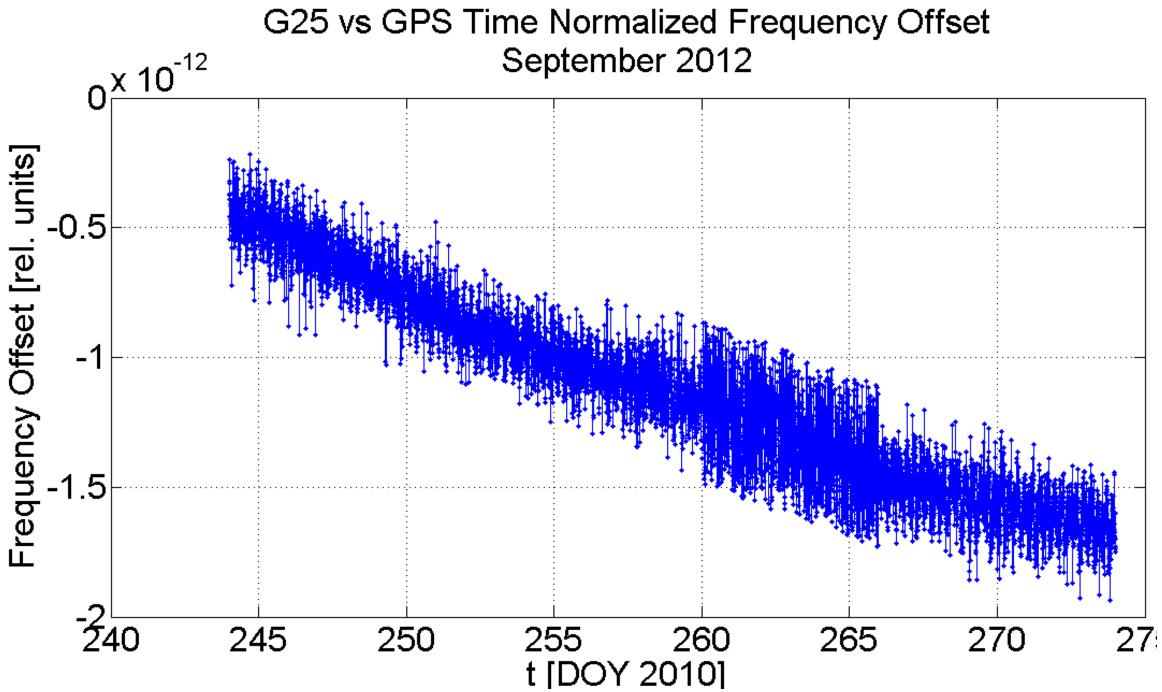
- *Nw* is the window length
- x_N is the phase signal in the window *Nw*
- τ_0 is the sampling time

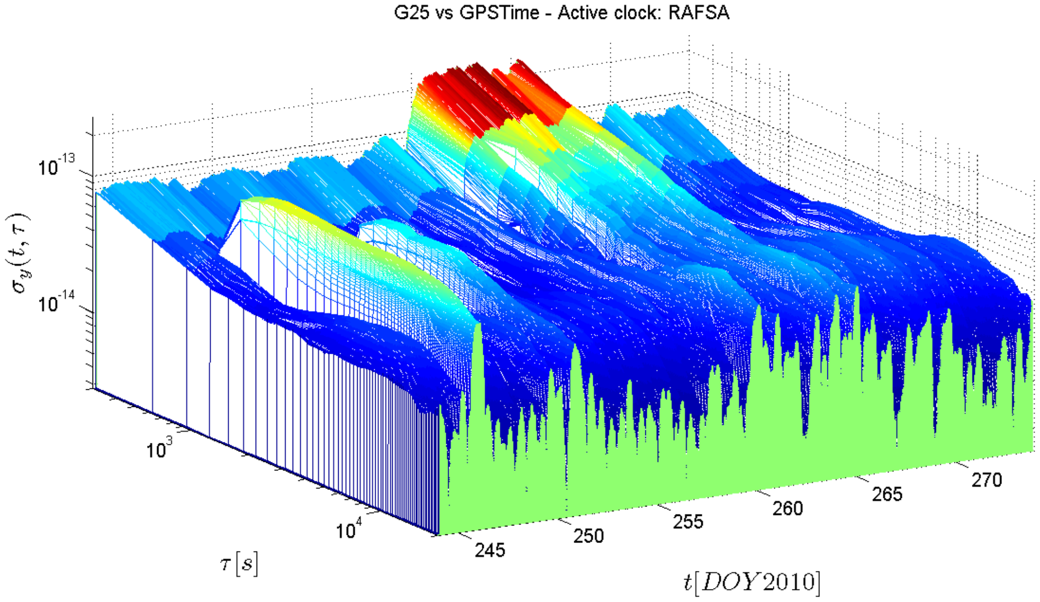
the DAVAR estimator

has no expectation value *E* because we have one realization only

L. Galleani, P. Tavella, "Dynamic Allan variance" , IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, UFFC, vol. 56, no. 3, March 2009, pp450-464
L. Galleani, P. Tavella, "The Dynamic Allan Variance V: Recent Advances in Dynamic Stability Analysis", EEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, Vol. 63, No. 4, Pag 624 - 635, April 2016

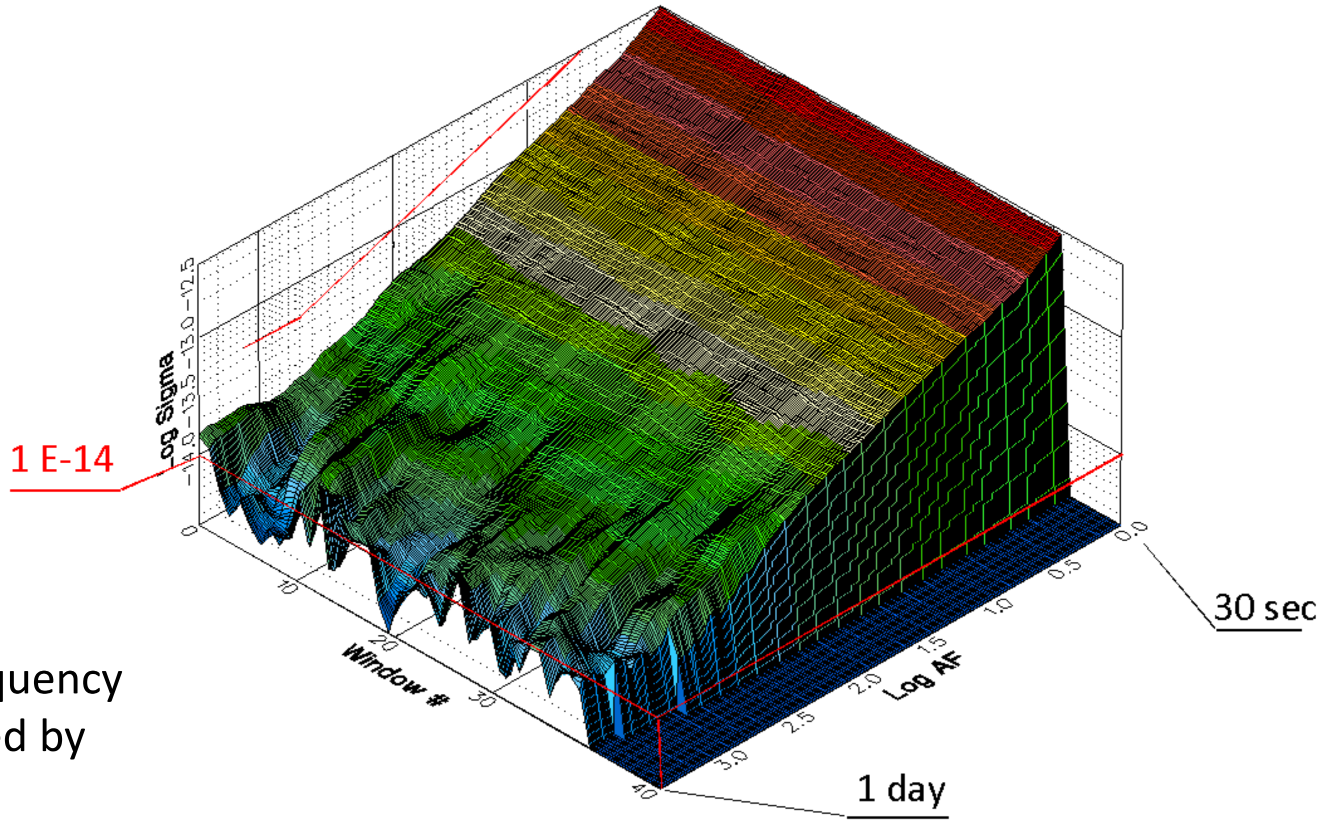
The change in instability is easily detected





The computation of the DADEV was available in STABLE 32

Demonstrating stationarity



DADEV
of a new space Rubidium atomic frequency standard, the Robust-RAFS, developed by Orolia Switzerland SA (SpectraTime).
The clock manufacturer demonstrates to the customer that the Robust-RAFS follows the specifications throughout the entire performance test.

(courtesy of Fabien Droz, Spectratime)

What we learnt

1. The (in)stability of time varying quantities may be highly impacting

2. Instability may be estimated by appropriate tools mathematical models including noises can be written

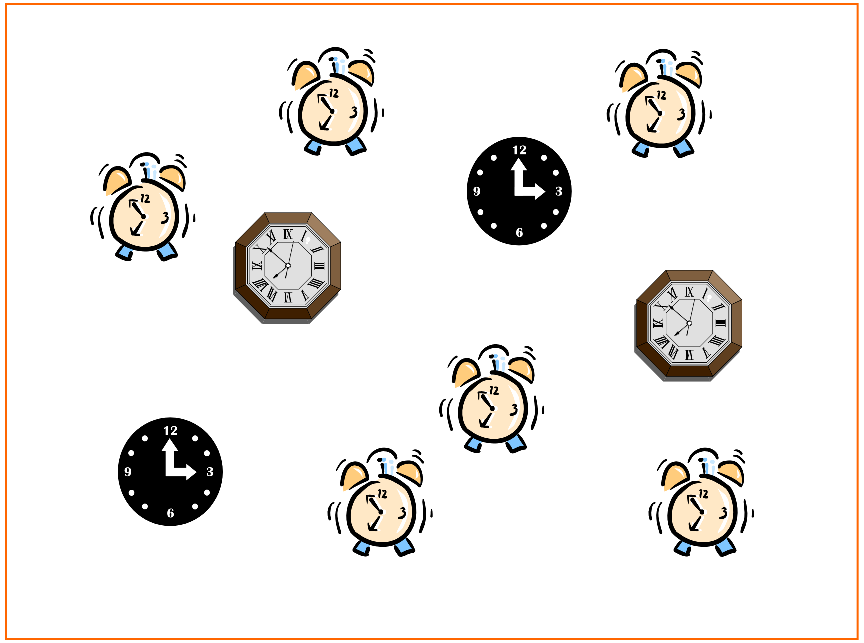
3. Models allow estimation, prediction, simulation, control

4. The behaviour may change due to ageing, failures, wearing… These changes are to be detected (rapidly) and the model is be dynamically updated

5. Some «preprocessing» on the data is fundamental to identify non typical behavious

How can an ensemble time be defined?

Clock ensemble algorithm



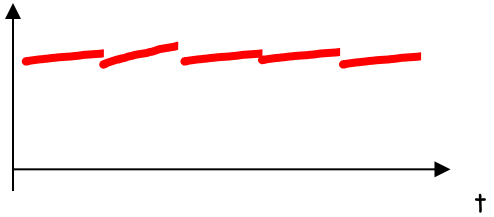
Optimising the contribution of each clock

Ensemble algorithms

Two main steps:

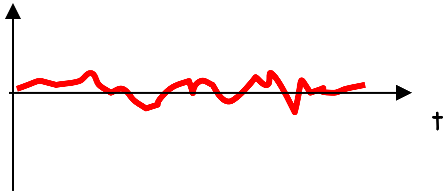
1. Average of clock reading

Stability



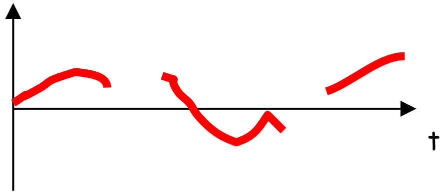
2. Steering

Accuracy

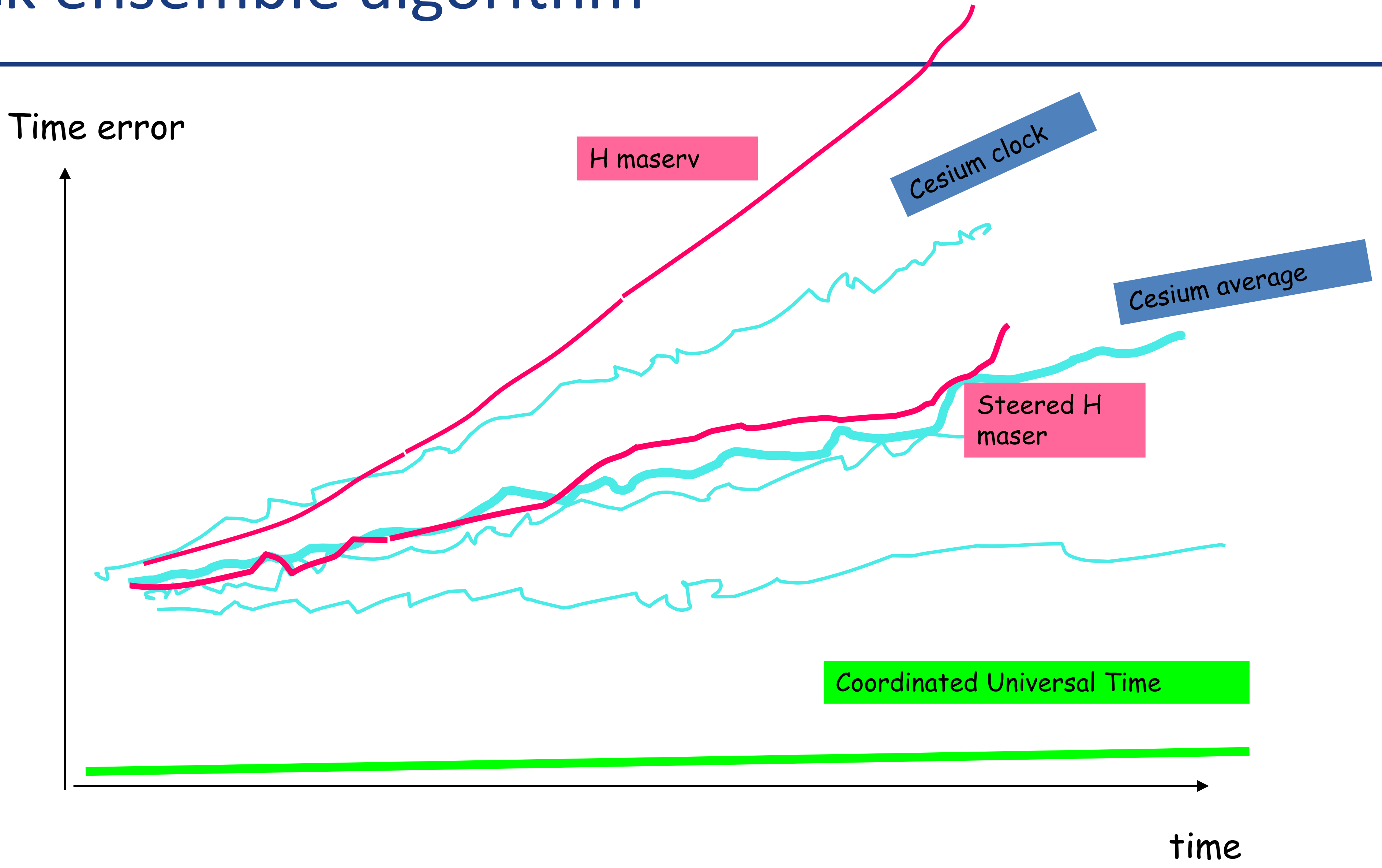


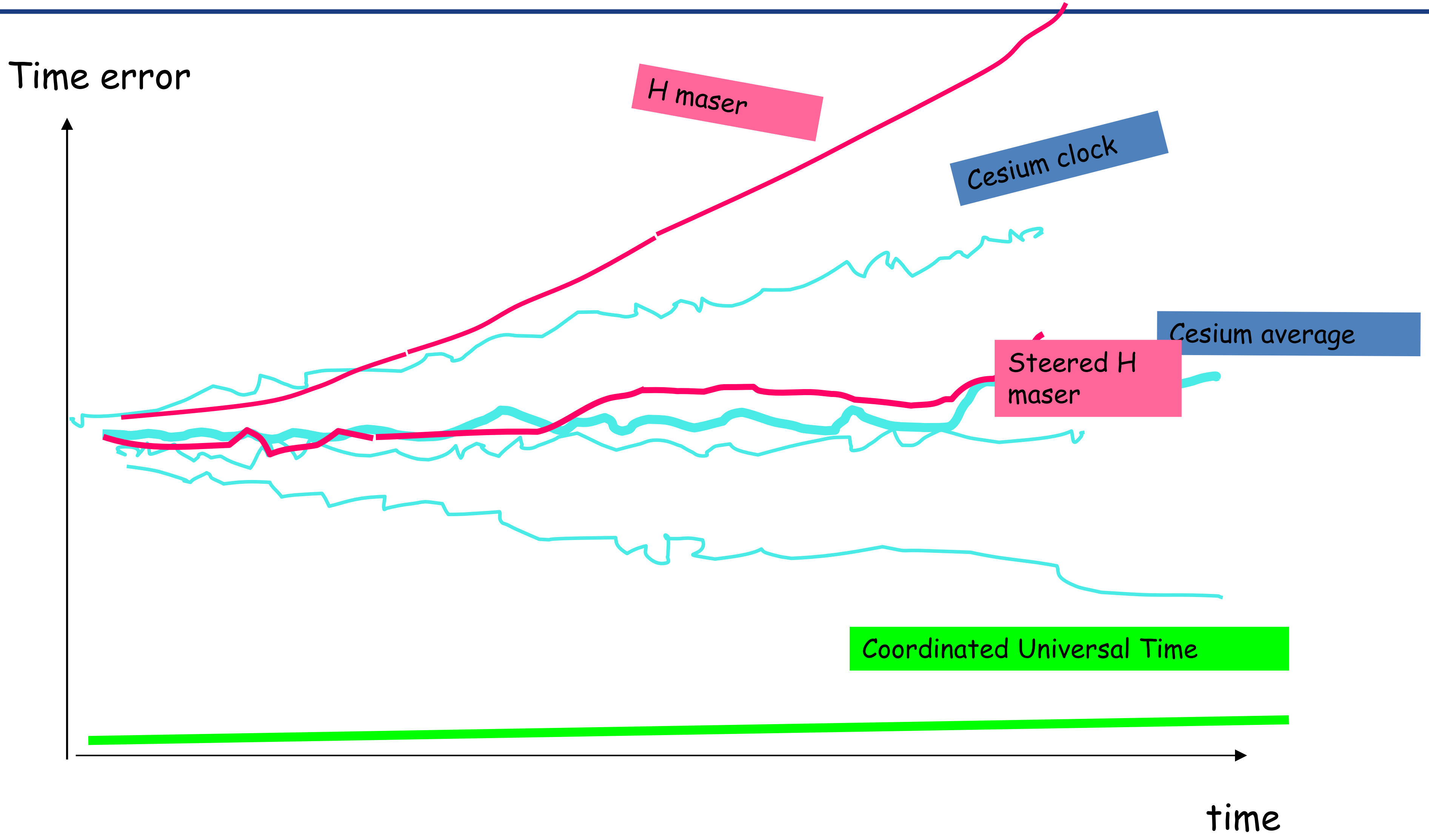
with care to

Reliability



Clock ensemble algorithm




Time error
H maser
Cesium clock
Cesium average
Steered H maser
Coordinated Universal Time
time

Ensemble algorithms

Weighted average time scale *TA*

$$TA = \frac{\sum_{i=1}^{N} p_i \left[h_i(t) \right]}{\sum_{i=1}^{N} p_i}$$

TA = ensemble time

h_i = reading of i-th clock

p_i= weight of i-th clock

$x_{ij} = h_j - h_i$ = measured clock difference

The reading h_i of i-th clock is not observable

$$TA = \frac{\sum_{i=1}^{N} p_i \left[h_i(t) \right]}{\sum_{i=1}^{N} p_i}$$

Only difference between clock reading are measurable

$x_{ij} = h_j - h_i$

The time scale *TA* is defined

through the difference versus each clock reading

$\boldsymbol{x_k = TA - h_k}$

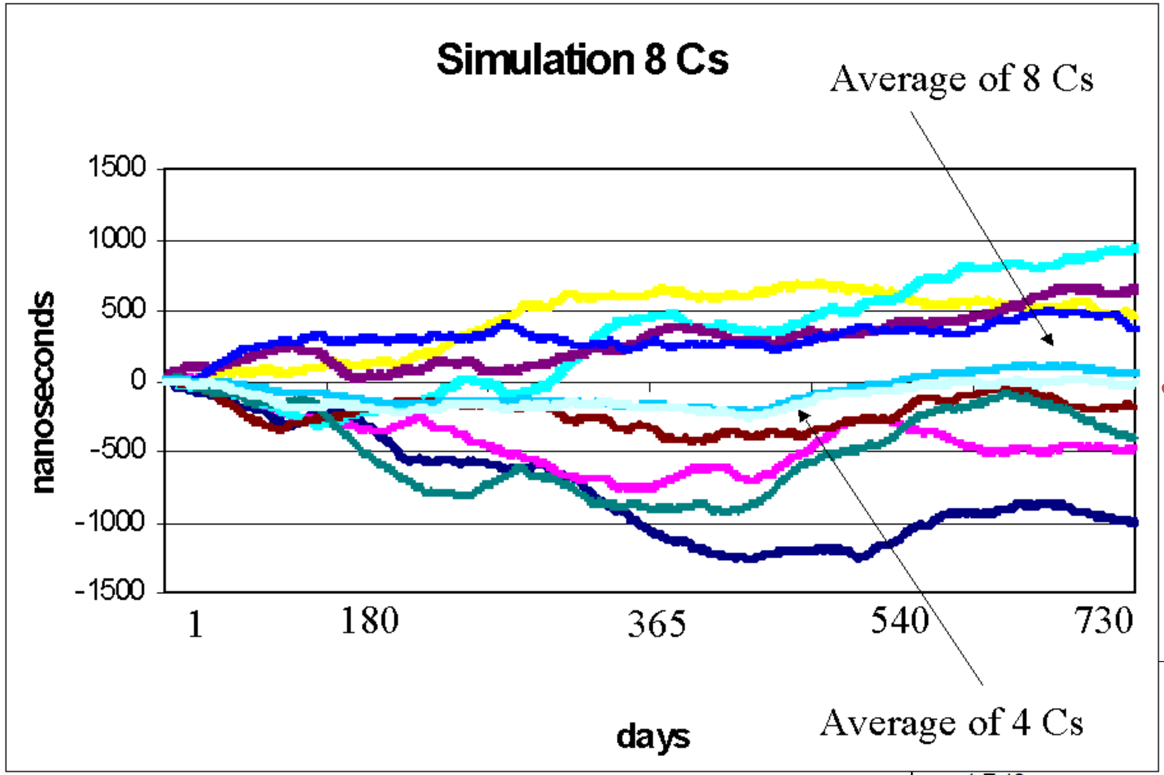
TA = ensemble time

h_i = reading of i-th clock

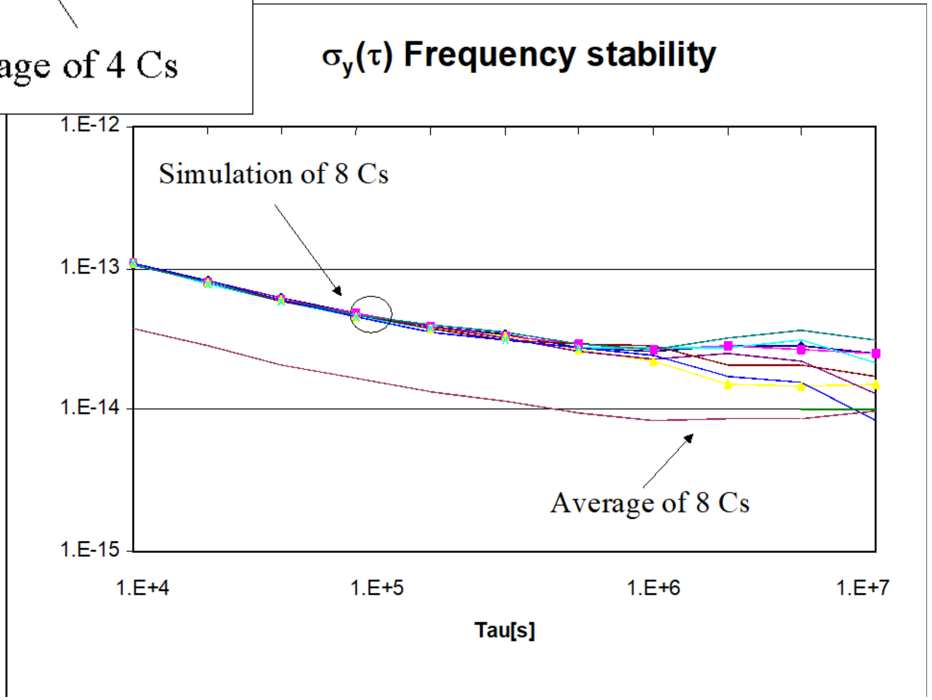
p_i= weight of i-th clock

$x_{ij} = h_j - h_i$ = measured clock difference

$$TA - h_k = \frac{\sum_{i=1}^{N} p_i \left[h_i(t) - h_k(t) \right]}{\sum_{i=1}^{N} p_i}$$



Stability improves



The average time scale TA is in the middle of clock readings
the weighted sum of distance between TA and clocks is zero

Weight

Ensemble Time Scale

$$TA - h_k = \frac{\sum_{i=1}^{N} p_i \left[h_i(t) - h_k(t) \right]}{\sum_{i=1}^{N} p_i}$$

weight p_i assignment

From the theory of least squares, the instability of *TA* is minimal if each clock is weighted as inversely proportional to its instability

$$p_i = \frac{1}{\sigma_y^2(\tau)}$$

in case of independent clock

τ

using complete clock covariance matrix **R**

$$p_i = \frac{\boldsymbol{R}^{-1} \cdot \boldsymbol{u}}{\boldsymbol{u}^T \cdot \boldsymbol{R}^{-1} \cdot \boldsymbol{u}}$$

in case of not independent clock

u is unit vector

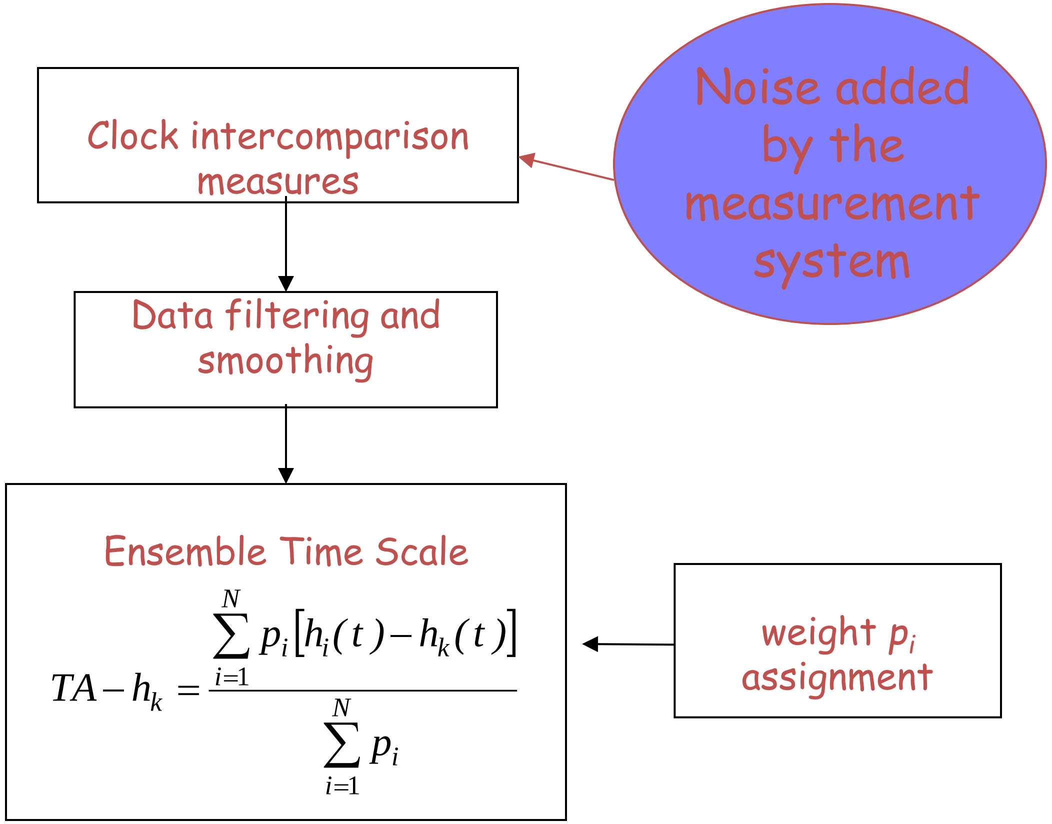
TA = ensemble time

h_i = reading of *i*-th clock

p_i= weight of *i*-th clock

$x_{ij} = h_j - h_i$ = measured clock difference

Ensemble algorithms

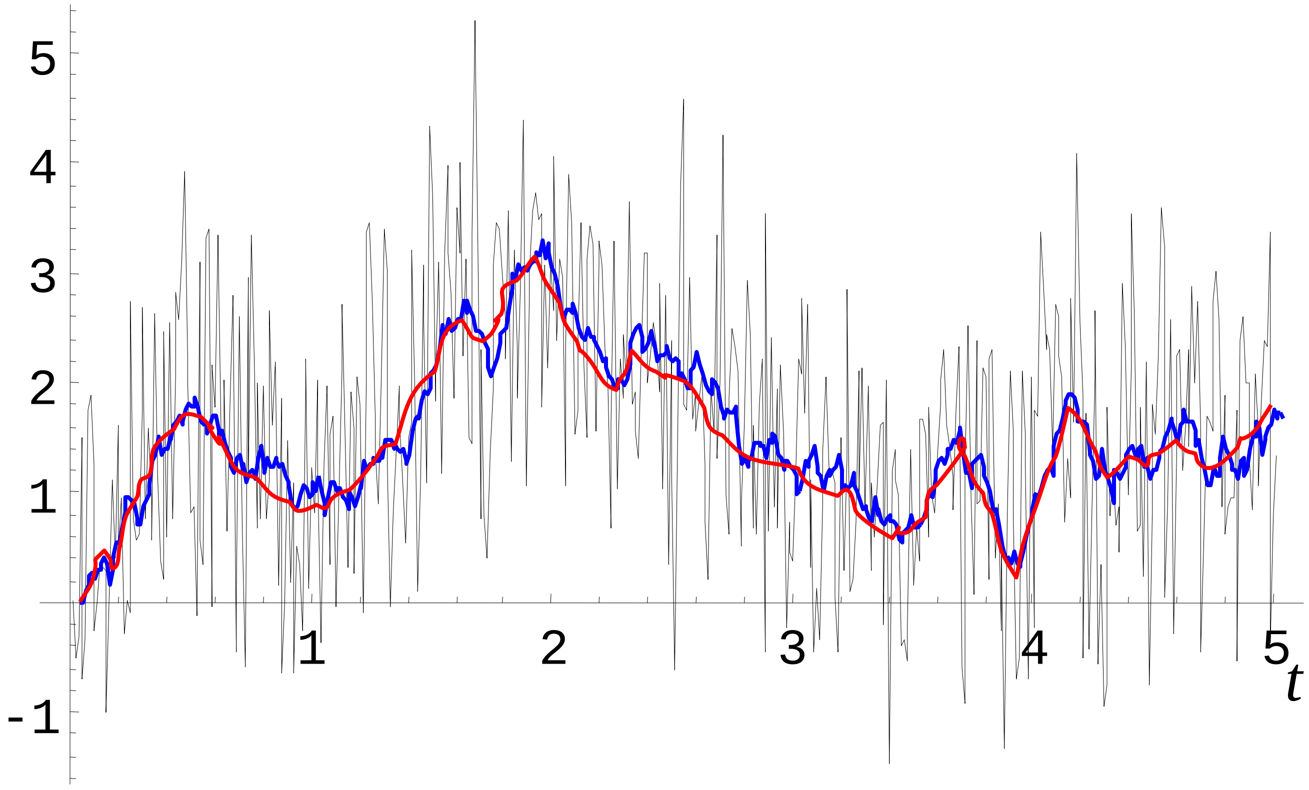


Data filtering and smoothing

Using a clock model and appropriate filtering techniques as

Kalman filter, AutoRegressive, Moving Average, ...

The noise added by the measurement system needs to be filtered by a smoothing technique



True clock random walk

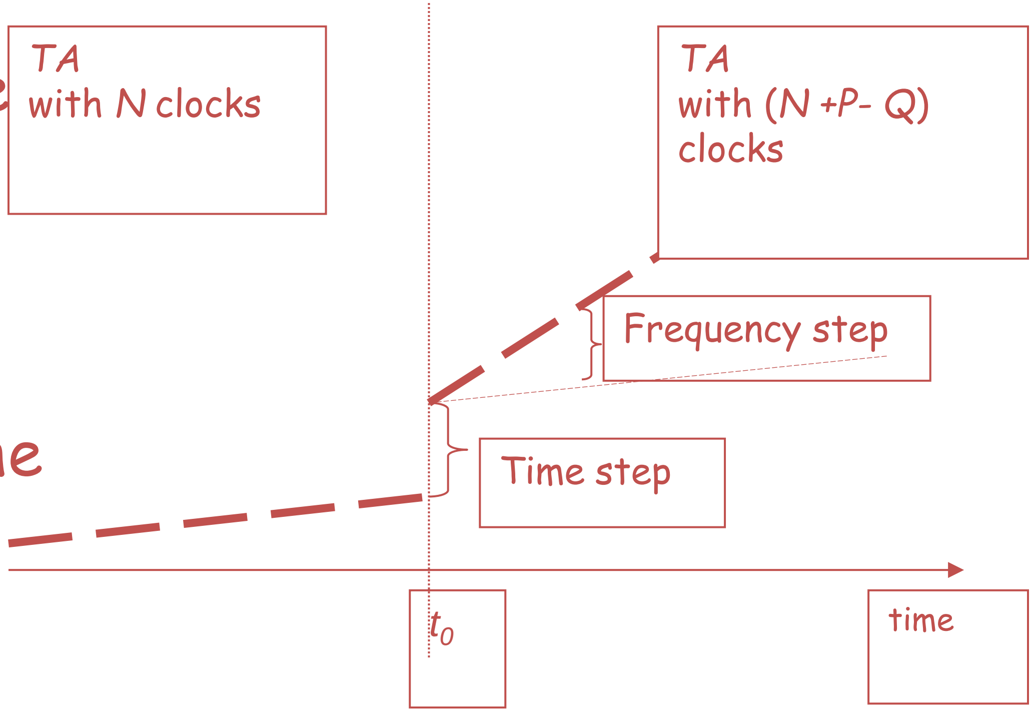
Noise added by the measurement system

Smoothed estimate

Stability asks for **insensitivity**

to changing number of clocks

or weights in the ensemble

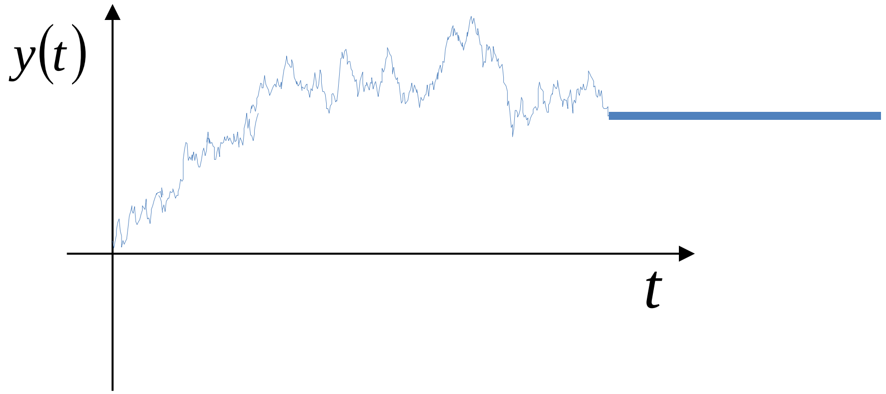


This means subtracting the **predictable** behaviour of the new (changed) clock

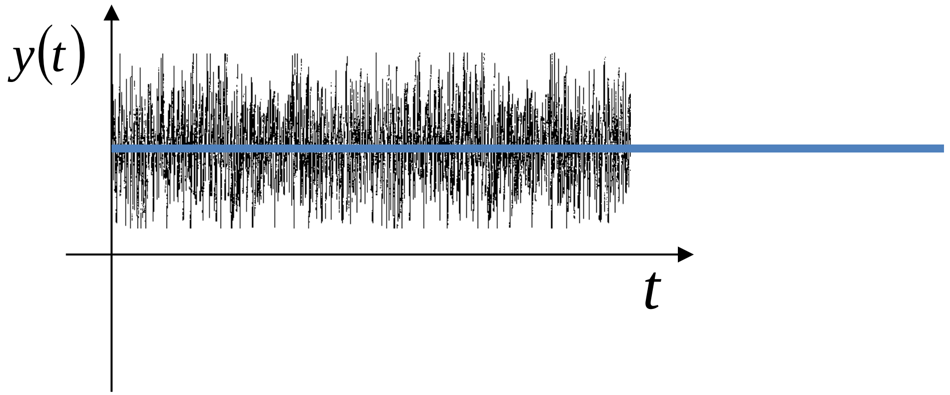
Prediction by means of clock model

Prediction

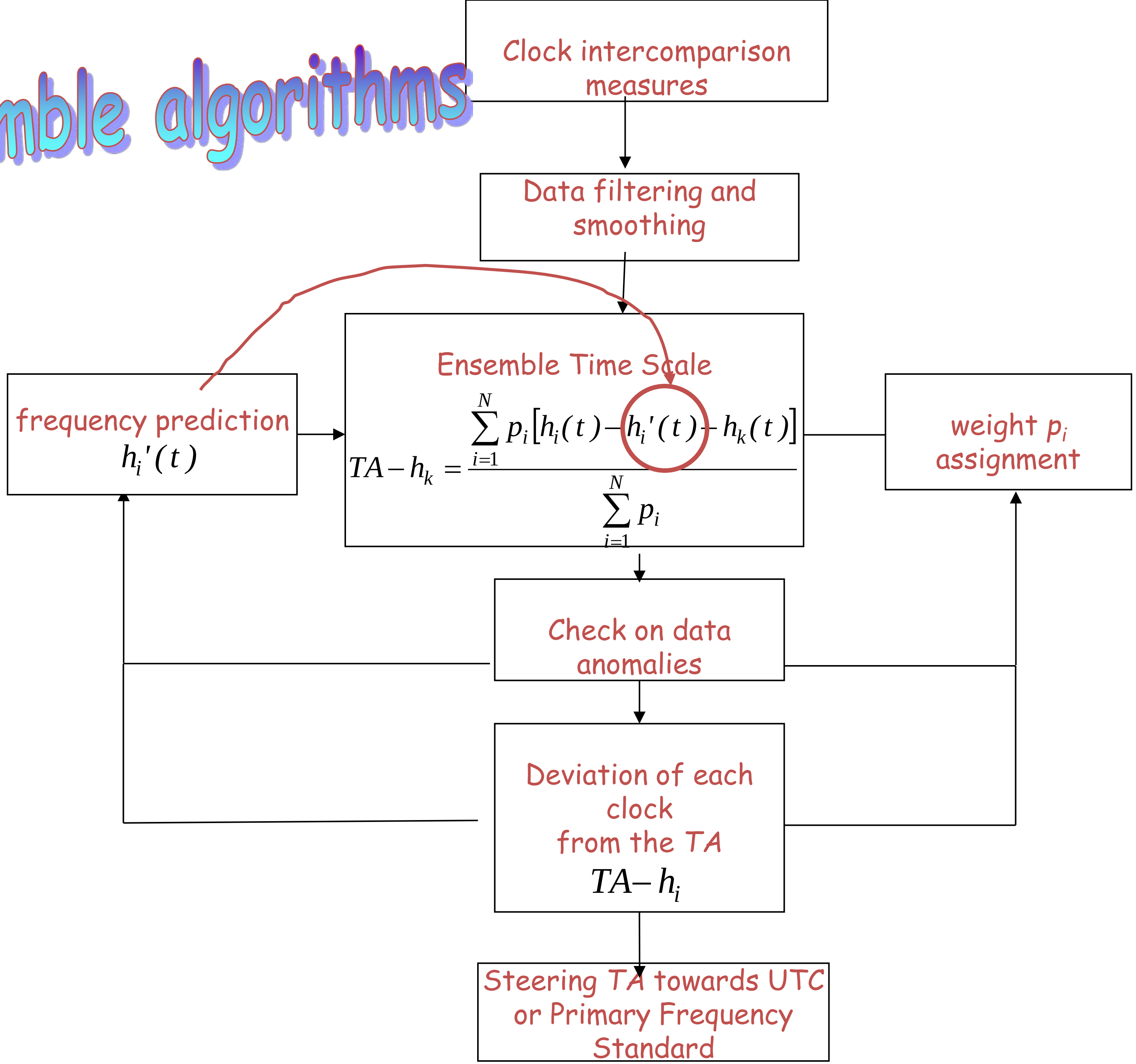
In case of Random Walk, the best prediction is the last value



In case of White Noise, the best prediction is the average value



Ensemble algorithms



Steering

versus UTC or primary frequency standard

From past measurement

the frequency of the ensemble time is **predicted**

(or the frequency of an individual clock is predicted)

In **real time** the predicted frequency offset is corrected estimated

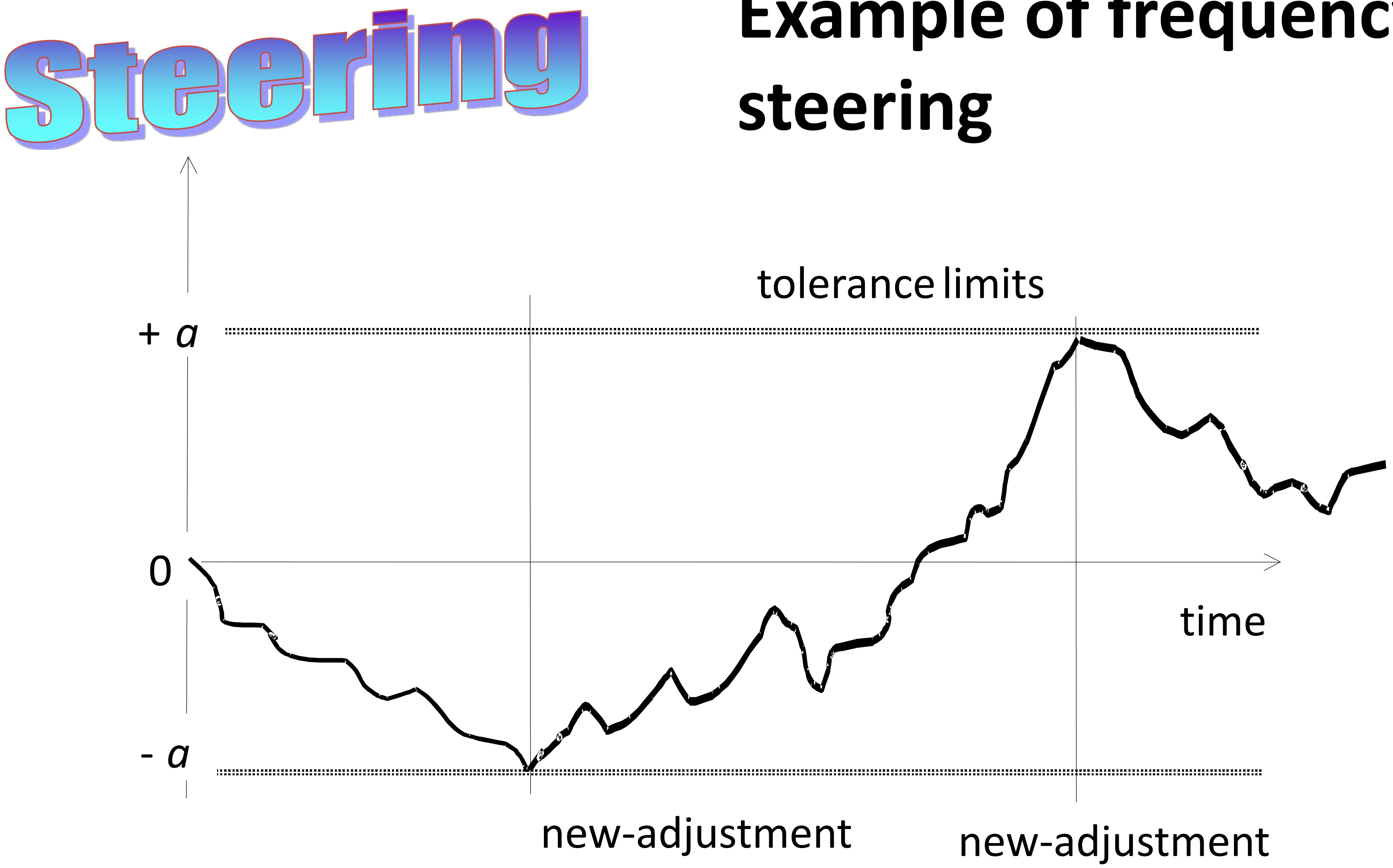
to generate a **Steered Ensemble Time**

(or to generate a steered individual clock)

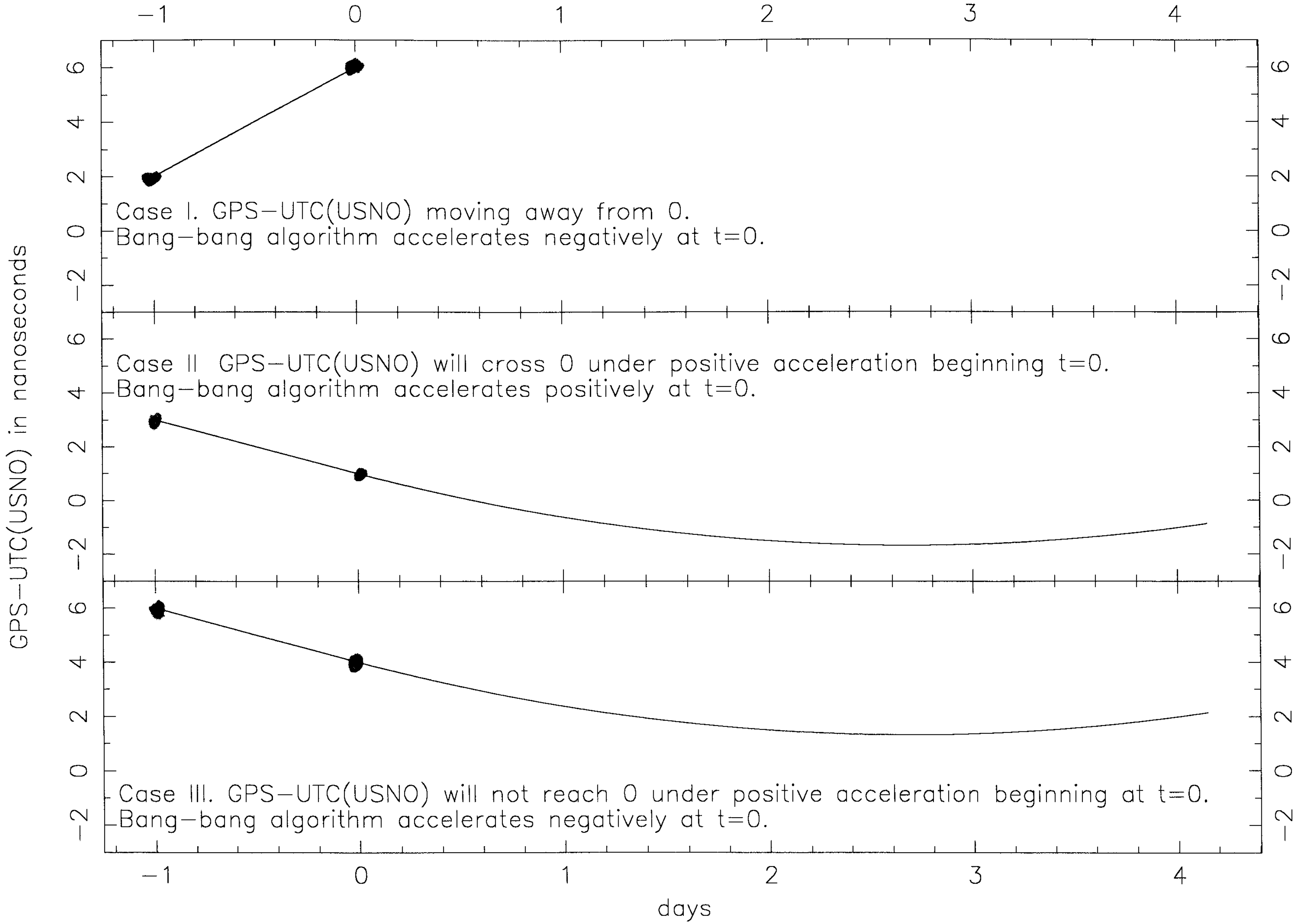
Example of phase adjustement

Steering

Example of frequency steering



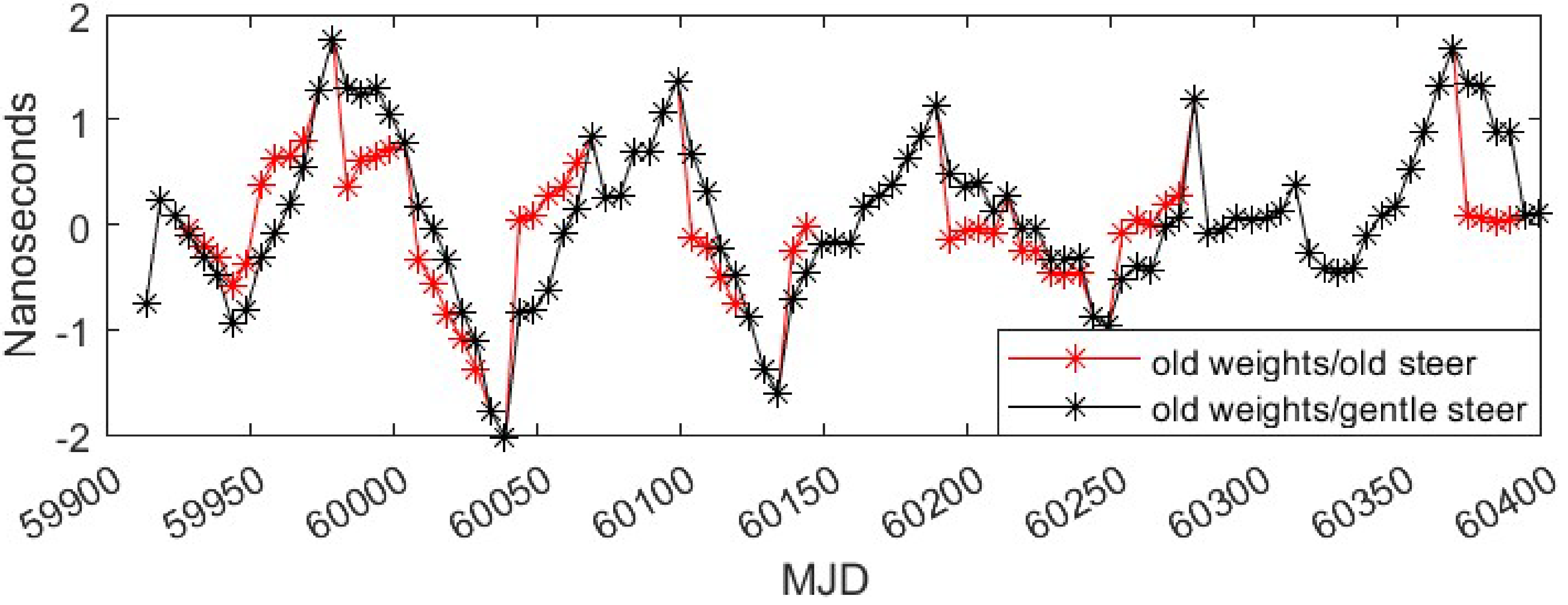
GPS “Bang-bang” steering the freq drift



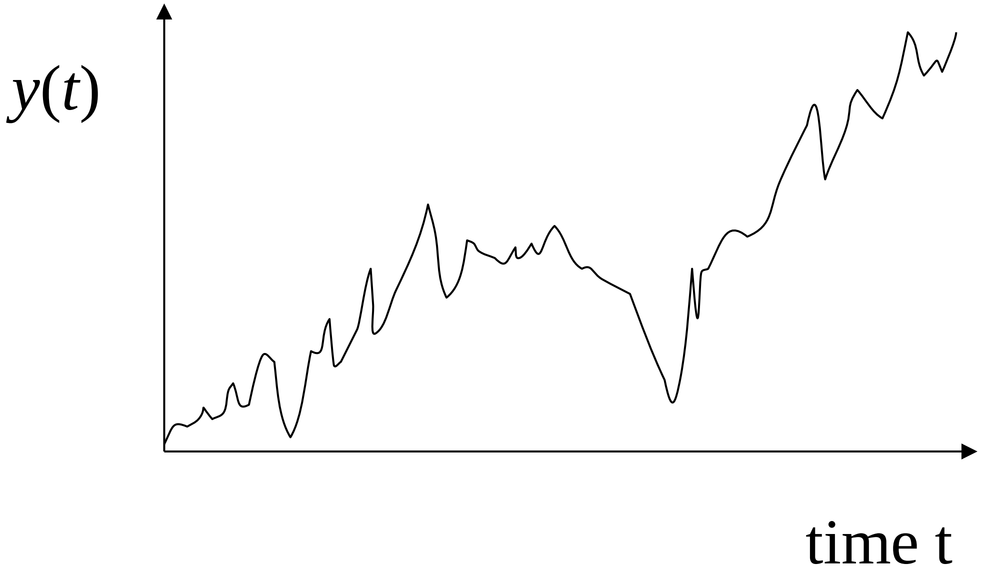
Steering

Example of the new steering algorithm of Rapid UTC

from time step to frequency steering



What about if we want to estimate a quantity changing by deterministic and random behaviour?



KALMAN filter:

Recursive version of least squares

Optimal linear estimation of a dynamical system with Gaussian noise

By Kalman - Bucy 1961

Kalman estimation needs:

1) a mathematical model of the evolution in time of the physical system + Gaussian noise

2) recursive measurement + Gaussian noise

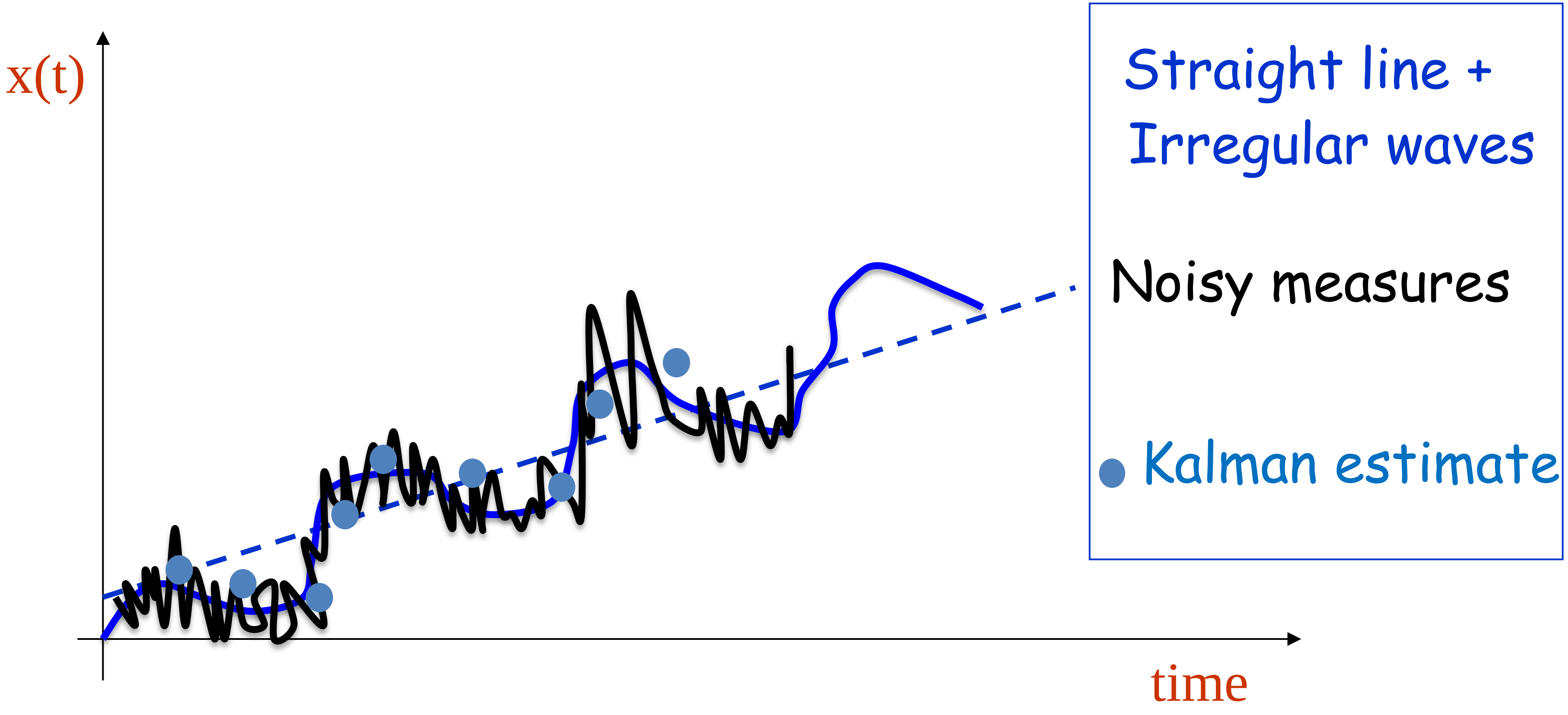
Gaussian noise:

$N(0, \Sigma)$

Example: ship position

Constant speed + waves $\quad x(t) = v\,t + \xi(t)$

Measures: $\quad z(t) = x(t)+\varepsilon(t)$



The Kalman filter mixes two information

1. from past measures and estimates at t_{k-1},

knowing the dynamics of the system

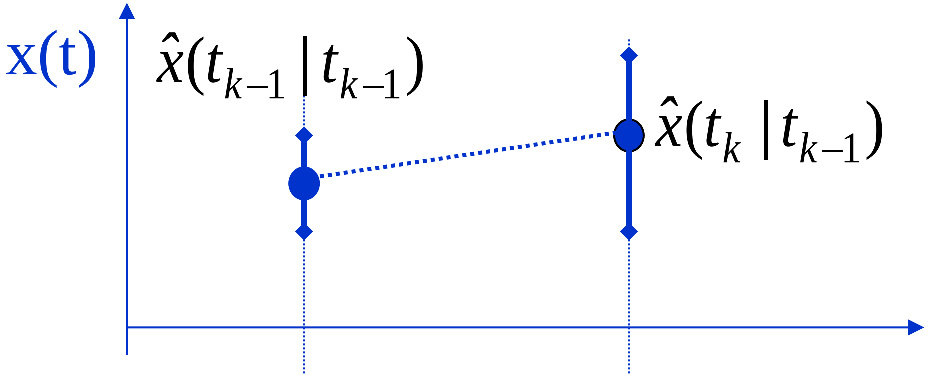
A prediction of the system at time t_k

is possible $\hat{x}(t_k \mid t_{k-1})$

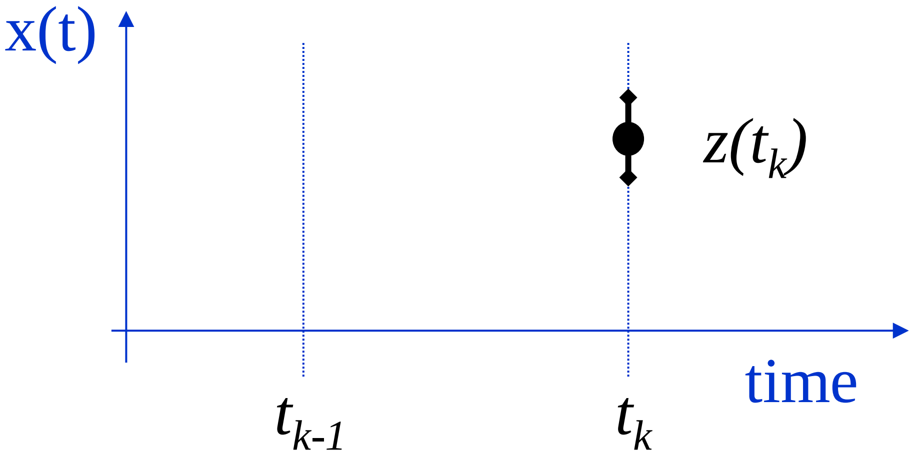
2. from the measurement system

A measure of the system at time t_k

is available $z(t_k)$



The prediction has an uncertainty due to the random disturbances and to the uncertainty of the previous estimate



The measure has noise and thus uncertainty

$$z(t) = x(t) + \varepsilon(t)$$

The two information are combined in a weighted average to obtain the best estimate at t_k

Estimate:

$$\hat{x}(t_k \mid t_k) = \text{weight}_{\text{prediction}} \; \hat{x}(t_k \mid t_{k-1}) + \text{weight}_{\text{measure}} \, z(t_k)$$

Uncertainty:

$$\Sigma_{\hat{x}}(t_k \mid t_k) = \text{combination of the prediction and measure uncertainty}$$

Thank you for listening till the end!

patrizia.tavella@bipm.org