Hadronic corrections to $(g-2)_\mu$ - status update of the Mainz workshop

Marc Vanderhaeghen

Fundamental Constants Meeting 2015, February 1-6, 2015

Hotel Frankenbach, Eltville, Germany

Hadronic contributions to the muon anomalous magnetic moment: strategies for improvements of the accuracy of the theoretical prediction

April 1–5, 2014 in Waldthausen Castle, Mainz, Germany

AND

\((g - 2)_\mu\): Quo vadis?

April 7–10, 2014 in Mainz, Germany

Mini Proceedings

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Mainz, Germany

ABSTRACT

We present the mini-proceedings of the workshops Hadronic contributions to the muon anomalous magnetic moment: strategies for improvements of the accuracy of the theoretical prediction and \((g - 2)_\mu\): Quo vadis? held in Mainz from April 1\(^{st}\) to 5\(^{th}\) and from April 7\(^{th}\) to 10\(^{th}\), 2014, respectively.

The web page of the conferences, which contains all talks, can be found at

- Hadronic contributions to the muon anomalous magnetic moment: https://indico.mitp.uni-mainz.de/conferenceDisplay.py?confId=13
- \((g - 2)_\mu\): Quo vadis?: https://indico.cern.ch/event/284012/
magnetic moment of muon: \((g-2)_\mu\)

- Magnetic moment

\[ \vec{m} = \mu_B \, g \vec{S} \]

- \(\mu_B\): Bohr magneton
- \(g\): gyromagnetic factor ~ 2

Dirac

- Anomalous part:

\[ a_\mu = \frac{(g-2)_\mu}{2} = \frac{\alpha_{em}}{2\pi} + ... = 0.00116... \]

Schwinger
magnetic moment of muon: \((g-2)_\mu\)

SM prediction for \(a_\mu\)

QED: \(a_\mu^{\text{QED}} = (11\,658\,471.896 \pm 0.008) \times 10^{-10}\)

up to \(O(\alpha_{em}^5)\)!

Aoyama, Hayakawa, Kinoshita, Nio (2012)

weak: \(a_\mu^{\text{weak}} = (15.4 \pm 0.1) \times 10^{-10}\)

strong: \(a_\mu^{\text{strong}} = (706.2 \pm 5.0) \times 10^{-10}\)

Hagiwara et al. (2011)

hadronic uncertainties completely dominate the accuracy of the SM result
(g-2)_μ : theory vs experiment

SM predictions for a_μ

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>JN 09 (e^+e^-)-based</td>
<td>-299 ± 65</td>
</tr>
<tr>
<td>DHMZ 10 (ν-based)</td>
<td>-195 ± 54</td>
</tr>
<tr>
<td>DHMZ 10 (e^+e^-)</td>
<td>-287 ± 49</td>
</tr>
<tr>
<td>HLMNT 11 (e^+e^-)</td>
<td>-261 ± 49</td>
</tr>
</tbody>
</table>

BNL-E821 (world average)

-200 -100 0 100 200 300 400 500 600 700

BNL-E821 measurement of a_μ

a_μ^{exp} = (11659208.9 ± 6.3) \times 10^{-10}

a_μ^{exp} - a_μ^{SM} = (26.1 ± 5.0^{th} ± 6.3^{exp}) \times 10^{-10}

Hagiwara et al. (2011)

3 - 4 σ deviation from SM value!

Errors or new physics?

New FNAL experiment (2016)

δa_μ^{FNAL} = 1.6 \times 10^{-10}

factor 4 improvement in exp. error

-> Improve theory!
strong contributions to $(g-2)_\mu$

Contributions from strong interactions NOT calculable within perturbative QCD

**Hadronic vacuum polarization**

$$a_{\mu}^{\text{l.o. had, VP}} = (694.9 \pm 4.3) \times 10^{-10}$$

Hagiwara et al. (2011)

Hadronic vacuum polarization determined by cross section measurements of $e^+e^- \rightarrow$ hadrons

**Hadronic light-by-light scattering**

$$a_{\mu}^{\text{had, LbL}} = (11.6 \pm 4.0) \times 10^{-10}$$

Jegerlehner, Nyffeler (2009)

Measurements of meson transition form factors required as input to reduce uncertainty
Optical theorem and analyticity allow to relate HVP contribution to \((g-2)_\mu\) with \(\sigma_{\text{had}} = \sigma(e^+e^- \rightarrow \text{hadrons})\)

\[
a^\text{had,VP}_\mu = \frac{1}{4\pi^3} \int_{4m^2_\pi}^{\infty} ds \, K(s) \, \sigma_{\text{had}}
\]

\(\sigma_{\text{had}}\): energy range up to 3 GeV essential!
measure $\sigma_{\text{had}}$ via ISR at BES-III

Approach for measuring hadronic cross section at modern particle factories with fixed c.m.s. energy $\sqrt{s}$:

*Initial State Radiation (ISR)*

ISR method allows access to mass range $M_{\text{hadr}} < 3$ GeV at BES-III

ongoing exp. program
HVP: most relevant channel $e^+ e^- \rightarrow \pi^+ \pi^-$

- KLOE and BABAR dominate the world average
- Relatively large systematic differences, esp. above $\rho$ peak
- Knowledge of $a_\mu^{\text{had}}$ dramatically limited due to this difference

Note: KLOE05 superseded by KLOE08
Status Hadronic Vacuum Polarization

Future improvement of $g_{\mu}^{\text{had}}$?

1\textsuperscript{st} priority:
Clarify situation regarding $\pi^+\pi^-$
(KLOE vs. BABAR puzzle)

2\textsuperscript{nd} priority:
Measure $3\pi$, $4\pi$ channels

3\textsuperscript{rd} priority:
KK and higher multiplicities
HVP: BES-III ISR results $e^+ e^- \rightarrow \pi^+ \pi^- \gamma$

Event yield after acceptance cuts only

Absolute cross section $\sigma(e+e^\rightarrow\pi^+\pi^-)$

2 normalization methods:
1) normalization to $L_{int}$ and Radiator-Function
2) normalization to $\mu\mu\gamma$, i.e. R ratio ($\pi\pi\gamma/\mu\mu\gamma$)

Cross section
Precision
luminosity / R ratio -1
= (0.35 ± 1.68) %

limited by low $\mu\mu\gamma$ statistics
HVP: progress in lattice QCD

Della Morte, Jäger, Jüttner, Wittig (2012)

status plot by L. Lellouche

lattice: 3-10 % quoted errors

experiment: 0.6 % errors
hadronic LbL scattering
hadronic LbL corrections to $(g-2)_\mu$

New FNAL and J-Parc $(g-2)_\mu$ expt.: $\delta a_{\mu}^{\exp} = 1.6 \times 10^{-10}$

$\begin{align*}
a_{\mu}^{\text{had, LbL}} &= (11.6 \pm 4.0) \times 10^{-10} \\
&\quad \text{Jegerlehner, Nyffeler (2009)} \\
a_{\mu}^{\text{had, LbL}} &= (10.5 \pm 2.6) \times 10^{-10} \\
&\quad \text{Prades, de Rafael, Vainshtein (2009)}
\end{align*}$

**experimental input:** meson transition FFs, $\gamma^* \gamma^* \rightarrow$ multi-meson states, meson Dalitz decays

BES-III, MAMI, ...

**theory developments:** models, sum rules, dispersion relations, lattice, ...
hadronic LbL corrections to \((g-2)_\mu\) : relevant contributions
\( \gamma \gamma \) physics: quark structure of mesons

\[ Q^2 = -Q^2 < 0 \]

spacelike

\[ Q^2 > 0 \]

timelike

\[ \gamma^* \gamma \rightarrow \pi^0 \]

\( F_{\pi^0}^0 \gamma^* \gamma \rightarrow \infty \)

\( 2 f_{\pi} \frac{Q^2}{Q^2} \)

BaBar and Belle data not in agreement!

paradigm for a whole program of hard processes planned at JLab12, Compass, EIC/ENC

transition region to perturbative QCD remains to be understood

B. Kloss (2011)
η and η′ transition form factors

Spacelike ($q^2 < 0$): $e^+e^-$ colliders

γ*γ → η

γ*γ → η'

Timelike ($q^2 > 0$): meson decays

π0, η, η'

γ* → γ e+ e−

γ* → γ e+ e−

Preliminary

new data from BES-III forthcoming

new MAMI/A2 data
The theory for sum rules for LbL scattering (I)

Helicity amplitudes:
\[ \gamma^*(\lambda_1, q_1) + \gamma^*(\lambda_2, q_2) \rightarrow \gamma^*(\lambda'_1, q_1) + \gamma^*(\lambda'_2, q_2) \]

Kinematical invariants:
\[ s = (q_1 + q_2)^2, \quad u = (q_1 - q_2)^2 \]
\[ \nu = \frac{s - u}{4}, \quad Q_1^2 \equiv -q_1^2, \quad Q_2^2 \equiv -q_2^2 \]

Helicity amplitudes:
\[ M_{\lambda'_1 \lambda'_2, \lambda_1 \lambda_2}(\nu, Q_1^2, Q_2^2) \quad \lambda = 0, \pm 1 \]

Discrete symmetries:
\[ P : M_{\lambda'_1 \lambda'_2, \lambda_1 \lambda_2} = M_{-\lambda'_1 - \lambda'_2, -\lambda_1 - \lambda_2} \]
\[ T : M_{\lambda'_1 \lambda'_2, \lambda_1 \lambda_2} = M_{\lambda_1 \lambda_2, \lambda'_1 \lambda'_2} \]

8 independent amplitudes:
\[ M_{+,+}, M_{+,-}, M_{++}, M_{00}, M_{+0}, M_{0+}, M_{++}, M_{00}, M_{0+}, 0 \]

T and L
sum rules for LbL scattering (II)

**Unitarity:** link to $\gamma^* \gamma^* \rightarrow X$ cross sections

$$W_{\lambda_1', \lambda_2', \lambda_1 \lambda_2} = \text{Im} \, M_{\lambda_1', \lambda_2', \lambda_1 \lambda_2}$$

$$\gamma^* \rightarrow X$$

$$\gamma^* \rightarrow X$$

$$W_{++,++} + W_{+-,+-} = 2\sqrt{X} \, (\sigma_0 + \sigma_2) = 2\sqrt{X} \, (\sigma_\| + \sigma_\perp) \equiv 4\sqrt{X} \, \sigma_{TT},$$

$$W_{++,+-} - W_{+-,++} = 2\sqrt{X} \, (\sigma_0 - \sigma_2) \equiv 4\sqrt{X} \, \tau_{TT},$$

$$W_{++,--} = 2\sqrt{X} \, (\sigma_\| - \sigma_\perp) \equiv 2\sqrt{X} \, \tau_{TT},$$

$$W_{00,00} = 2\sqrt{X} \, \sigma_{LL},$$

$$W_{+,0+} = 2\sqrt{X} \, \sigma_{LT},$$

$$W_{+,0+} + W_{0+,-0} = 4\sqrt{X} \, \tau_{TL},$$

$$W_{++,00} - W_{0+,0-} = 4\sqrt{X} \, \tau_{TL}.$$
Sum rules for LbL scattering (III)

Imaginary part of the amplitude - photon-photon fusion into leptons and hadrons:

\[
\text{Im} f^-(s) = -\frac{s}{8\pi}\left[\sigma_2(s) - \sigma_0(s)\right] \\
\text{Im} f^+(s) = -\frac{s}{8\pi}\left[\sigma_{tot}(s)\right]
\]

\[
\int_{s_0}^{\infty} \frac{ds}{s} \left[\sigma_2(s) - \sigma_0(s)\right] = 0
\]

Real part of the amplitude - low-energy structure of the elastic LbL scattering:

\[
L^{(8)} = c_1(F_{\mu\nu}F^{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}^{\mu\nu})^2
\]

Euler, Heisenberg (1936)

\[
c_1 \pm c_2 = \frac{1}{8\pi} \int_{s_0}^{\infty} \frac{ds}{s^2} \left[\sigma_{\parallel}(s) \pm \sigma_{\perp}(s)\right]
\]
sum rules for LbL scattering (IV)

3 superconvergent relations:

- Helicity difference sum rule
- Sum rules involving longitudinal photons

SRs involving LbL low-energy constants:

\[ c_1 \pm c_2 = \frac{1}{8\pi} \int_{s_0}^{\infty} \frac{ds}{s^2} [\sigma_\parallel(s) \pm \sigma_\perp(s)] \]

Pascalutsa, Pauk, Vdh (2012)

for \( Q^2 = 0 \): GDH sum rule

Gerasimov, Moulin (1975), Brodsky, Schmidt (1995)

+ 6 new LECs at next order

Sum rules have been tested in perturbative QFT both at tree-level and 1-loop level
single meson production in $\gamma\gamma$ collisions (I)

- two-photon state: produced meson has $C=+1$
- both photons are real: $J=1$ final state is forbidden (Landau-Yang theorem);
the main contribution comes from
$J=0$: $0^+$ (pseudoscalar) and $0^{++}$ (scalar)
and $J=2$: $2^{++}$ (tensor)

- the SRs hold separately for channels of given intrinsic quantum numbers: isoscalar and isovector mesons, $c\bar{c}$ states

- input for the absorptive part of the SRs: $\gamma\gamma$-hadrons response functions, can be expressed in terms of $\gamma\gamma\rightarrow M$ transition form factors

\[\sigma_{\gamma\gamma\rightarrow M}^{\chi}(s) \approx (2J + 1)16\pi^2 \frac{\Gamma_{\gamma\gamma}}{m_M} \delta(s - m_M^2)\]

\[\Gamma_{\gamma\gamma}(\mathcal{P}) = \frac{\pi\alpha^2}{4} m^3 |F_{M\gamma*\gamma*}(0, 0)|^2\]

meson contribution to the cross-section in the narrow-resonance approximation

two-photons decay rate for the meson
single meson production in $\gamma\gamma$ collisions (II)

**the I=0 channel**

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\int \frac{d\sigma}{s} (\sigma_2 - \sigma_0)$ [nb]</th>
<th>$c_1$ [10^{-4} GeV^{-4}]</th>
<th>$c_2$ [10^{-4} GeV^{-4}]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>$-191 \pm 10$</td>
<td>0</td>
<td>$0.65 \pm 0.03$</td>
</tr>
<tr>
<td>$\eta'$</td>
<td>$-300 \pm 10$</td>
<td>0</td>
<td>$0.33 \pm 0.01$</td>
</tr>
<tr>
<td>$f_0(980)$</td>
<td>$-19 \pm 5$</td>
<td>$0.020 \pm 0.005$</td>
<td>0</td>
</tr>
<tr>
<td>$f'_0(1370)$</td>
<td>$-91 \pm 36$</td>
<td>$0.049 \pm 0.019$</td>
<td>0</td>
</tr>
<tr>
<td>$f_2(1270)$</td>
<td>$449 \pm 52$</td>
<td>$0.141 \pm 0.016$</td>
<td>0.141 $\pm 0.016$</td>
</tr>
<tr>
<td>$f'_2(1525)$</td>
<td>$7 \pm 1$</td>
<td>$0.002 \pm 0.000$</td>
<td>$0.002 \pm 0.000$</td>
</tr>
<tr>
<td>$f_2(1565)$</td>
<td>$56 \pm 11$</td>
<td>$0.012 \pm 0.002$</td>
<td>$0.012 \pm 0.002$</td>
</tr>
<tr>
<td>Sum</td>
<td>$-89 \pm 66$</td>
<td>$0.22 \pm 0.03$</td>
<td>$1.14 \pm 0.04$</td>
</tr>
</tbody>
</table>

dominant contribution to $c_2$ comes from $\eta$, $\eta'$ and $f_2(1270)$

dominant contribution to $c_1$ comes from $f_2(1270)$

**the I=1 channel**

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\int \frac{d\sigma}{s} (\sigma_2 - \sigma_0)$ [nb]</th>
<th>$c_1$ [10^{-4} GeV^{-4}]</th>
<th>$c_2$ [10^{-4} GeV^{-4}]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0$</td>
<td>$-195 \pm 13$</td>
<td>0</td>
<td>$10.94 \pm 0.70$</td>
</tr>
<tr>
<td>$a_0(980)$</td>
<td>$-20 \pm 8$</td>
<td>$0.021 \pm 0.007$</td>
<td>0</td>
</tr>
<tr>
<td>$a_2(1320)$</td>
<td>$134 \pm 8$</td>
<td>$0.039 \pm 0.002$</td>
<td>$0.039 \pm 0.002$</td>
</tr>
<tr>
<td>$a_2(1700)$</td>
<td>$18 \pm 3$</td>
<td>$0.003 \pm 0.001$</td>
<td>$0.003 \pm 0.001$</td>
</tr>
<tr>
<td>Sum</td>
<td>$-63 \pm 17$</td>
<td>$0.06 \pm 0.01$</td>
<td>$10.98 \pm 0.70$</td>
</tr>
</tbody>
</table>

dominant contribution to $c_2$ comes from $\pi^0$

Pascalutsa, Pauk, Vdh (2012)
single meson production in $\gamma\gamma$ collisions (III)

- one photon is virtual $Q_1^2$, second is quasi-real $Q_2^2 \approx 0$
- axial-vector mesons $1^{++}$ are allowed
- $f_1(1285), f_1(1420)$ transition FFs constrained from LEP (L3) data

| $m_M$ [MeV] | $\Gamma_{\gamma\gamma}$ [keV] | $\int \frac{ds}{s^2} \sigma_{\parallel}(s)$ [nb / GeV$^2$] | $\int ds \frac{1}{s^2} \frac{\tau_{1\parallel}^{f_1}}{Q_1 Q_2} |Q_1^2=0$ [nb / GeV$^2$] | $\int ds \frac{1}{s^2} \sigma_{\parallel} + \frac{1}{s^2} \frac{\tau_{1\parallel}^{f_1}}{Q_1 Q_2} |Q_1^2=0$ [nb / GeV$^2$] |
|---|---|---|---|---|
| $f_1(1285)$ | $1281.8 \pm 0.6$ | $3.5 \pm 0.8$ | $0$ | $-93 \pm 21$ |
| $f_1(1420)$ | $1426.4 \pm 0.9$ | $3.2 \pm 0.9$ | $0$ | $-50 \pm 14$ |
| $f_0(980)$ | $980 \pm 10$ | $0.29 \pm 0.07$ | $20 \pm 5$ | $20 \pm 5$ |
| $f_0'(1370)$ | $1200 - 1500$ | $3.8 \pm 1.5$ | $48 \pm 19$ | $48 \pm 19$ |
| $f_2(1270)$ | $1275.1 \pm 1.2$ | $3.03 \pm 0.35$ | $138 \pm 16$ | $138 \pm 16$ |
| $f_2'(1525)$ | $1525 \pm 5$ | $0.081 \pm 0.009$ | $1.5 \pm 0.2$ | $1.5 \pm 0.2$ |
| $f_2(1565)$ | $1562 \pm 13$ | $0.70 \pm 0.14$ | $12 \pm 2$ | $12 \pm 2$ |
| Sum | | | | $76 \pm 36$ |

sum rules allow to constrain so far unmeasured contributions, e.g. $\gamma^* \gamma^* \rightarrow$ tensor mesons

Using data for $\eta, \eta'$ FF

Using data for $f_1(1285), f_1(1420)$ FF

$f_2(1270)$
single meson contributions to $a_\mu$ (I)

- axial-vector meson contribution to $a_\mu$ re-evaluated
- Landau-Yang theorem constraint implemented
- $f_1(1285), f_1(1420)$ transition FFs from L3 data

Pauk, Vdh (2013)

$$a_\mu^{LbL} = \frac{\alpha}{(2\pi)^2} \frac{\Lambda_{A1}^6 \Lambda_{A2}^6 \tilde{\Gamma}_{\gamma\gamma}(A)}{m m_A^5} \int dQ_1 \int dQ_2 \left[ 2w_a(Q_1, Q_2) + w_c(Q_1, Q_2) \right]$$
single meson contributions to $a_\mu$ (II)

axial-vector meson re-evaluation was reported in 2 works
- implementation of Landau-Yang theorem constraint leads to difference with previous results

tensor mesons evaluated for first time

<table>
<thead>
<tr>
<th></th>
<th>pseudo-scalars</th>
<th>axial-vectors</th>
<th>scalars</th>
<th>tensors</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPP</td>
<td>85 ± 13</td>
<td>2.5 ± 1.0</td>
<td>−7 ± 2</td>
<td>-</td>
</tr>
<tr>
<td>HKS</td>
<td>82.7 ± 6.4</td>
<td>1.7 ± 1.7</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>MV</td>
<td>114 ± 10</td>
<td>22 ± 5</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>KN</td>
<td>83 ± 12</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>J</td>
<td>93.9 ± 12.4</td>
<td>~ 7</td>
<td>−6.0 ± 1.2</td>
<td>-</td>
</tr>
<tr>
<td>this work</td>
<td>-</td>
<td>6.4 ± 2.0</td>
<td>(0.9 ~ 3.1) ± 0.8</td>
<td>1.1 ± 0.1</td>
</tr>
</tbody>
</table>

total $(6.6 ~ 4.4) \pm 2.9 \times 10^{-11}$
multi-meson production in $\gamma\gamma$ collisions (I)

new estimate for pion loop contribution (with full VMD FF)  
Bijnens (2014)

$a_\mu^{\text{LbL}} \pi$-loop = ($-2.0 \pm 0.5$) x 10^{-10}  
integrating momenta in loop up to 1 GeV

contribution of multi-meson channels  
Dai, Pennington (2014)

sum rules may be used as a consistency check of models
multi-meson production in $\gamma\gamma$ collisions (II)

- new dispersion formalism for $\gamma^* \gamma^* \rightarrow \pi \pi$

  - $Q_1^2 = 0.5 \text{ GeV}^2$, $Q_2^2 = 0$
  - Moussallam (2013)
  - Hoferichter, Colangelo, Procura, Stoffer (2013)
  - Assmusen, Masjuan, Vdh (2014)

- forthcoming BES-III data for $\gamma^* \gamma^* \rightarrow \pi \pi$
  - Guo (2014)
dispersion relation approaches for $a_\mu$ (I)

Hoferichter, Colangelo, Procura, Stoffer (2014)

**dispersion formalism for $\gamma^* \gamma^* \rightarrow \gamma^* \gamma$**

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} + \Pi_{\mu\nu\lambda\sigma}^\ast + \cdots \]

**master formula for $a_\mu$**

\[ a_{\mu}^{\pi\pi} = e^6 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{l_{\pi\pi}}{q_1^2 q_2^2 s ((p + q_1)^2 - m^2)((p - q_2)^2 - m^2)} , \]

\[ l_{\pi\pi} = \sum_{i \in \{1,2,3,6,14\}} \left( T_{i,s} l_{i,s} + 2 T_{i,u} l_{i,u} \right) + 2 T_{9,s} l_{9,s} + 2 T_{9,u} l_{9,u} + 2 T_{12,u} l_{12,u} \]

with $l_{i,(s,u)}$ dispersive integrals and $T_{i,(s,u)}$ integration kernels

\[ l_{1,s} = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s' - s} \left( \frac{1}{s' - s} - \frac{s' - q_1^2 - q_2^2}{\lambda(s', q_1^2, q_2^2)} \right) \text{Im} \bar{h}^0_{++,-,-}(s'; q_1^2, q_2^2; s, 0) , \]

\[ l_{6,s} = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{(s' - q_1^2 - q_2^2)(s' - s)^2} \text{Im} \bar{h}^2_{+-,-,-}(s'; q_1^2, q_2^2; s, 0) \left( \frac{75}{8} \right) \]

Helicity amplitudes contribute up to $J = 2$ ($S$ and $D$ waves)
dispersion relation approaches for $a_\mu$ (II)

**dispersion formalism** directly for $a_\mu$  

**Pauk, Vdh (2014)**

$$a_\mu = \lim_{k \to 0} F_2(k^2, (p + k)^2, p^2)$$

$$F_2(0) = \frac{1}{2\pi i} \int \frac{dk^2}{k^2} \text{Abs} \ F_2(k^2)$$

$$a_\mu = F_2(0)$$

$$F_2(k^2) = e^6 \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda} (-1)^{\lambda + \lambda_1 + \lambda_2 + \lambda_3} \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} L_{\lambda_1 \lambda_2 \lambda_3 \lambda} (p, p', q_1, k - q_1 - q_2, q_2)$$

**analytic structure**

$$\Pi_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (q_1, q_2; q_3) = \epsilon^\mu (q_1, \lambda_1) \epsilon^{\nu} (q_2, \lambda_2) \epsilon^{\lambda} (q_3, \lambda_3) \epsilon^{\rho} (q_4, \lambda_4) \Pi_{\mu \nu \lambda \rho} (q_1, q_2; q_3)$$

**weighting functions (entire)**

$$\frac{\Pi_{\lambda_1 \lambda_2 \lambda_3 \lambda} (q_1, k - q_1 - q_2, q_2, k)}{q_1^2 q_2^2 (k - q_1 - q_2)^2 [(p + q_1)^2 - m^2] [(p + k - q_2)^2 - m^2]}$$
dispersion relation approaches for $a_\mu$ (III)

**Proof of principle:** pole contributions

**2-particle cuts**

$\pi^0$ pole at $(q_i + q_j)^2 = M^2$

Analytical structure of LbL amplitude

Vector - poles at $q_i^2 = \Lambda^2$

2-particle discontinuities
dispersion relation approaches for $a_\mu$ (IV)

3-particle discontinuities
dispersion relation approaches for $a_\mu$ (V)

reconstruction of $a_\mu$ from dispersion integral: proof of principle

$$F_2^{(i)}(0) = \frac{1}{2\pi i} \int_{M^2}^{\infty} \frac{dt}{t} \text{Disc}_t^{(2)} F_2^{(i)}(t) + \frac{1}{2\pi i} \int_{0}^{\infty} \frac{dt}{t} \text{Disc}_t^{(3)} F_2^{(i)}(t)$$

$a_\mu \ast M^3/(\alpha \Gamma_{\gamma\gamma})$ (in GeV$^2$): diagram a

$\Lambda = 0.77$ GeV

Pauk, Vdh (2014)
**Summary and outlook**

- **HVP**: new experimental program at BES-III
  - first results for $e^+ e^- \rightarrow \pi^+ \pi^- \gamma$
  - aim: hadronic cross sections to 1% accuracy
  - strong lattice effort to rival experimental accuracy

- **HLbL**: new theoretical tools for $\gamma^* \gamma^* \rightarrow X$
  - sum rules, dispersive frameworks for transition FFs:
    - allow to include experimental constraints
  - new evaluation of heavier meson contributions: $a_\mu = (6.6 \sim 4.4) \pm 2.9 \times 10^{-11}$
  - pioneering lattice efforts

- **new dispersion relation** frameworks for HLbL to $a_\mu$:
  - require close collaboration with experiment (spacelike, timelike, meson decays)

- Outcome of Mainz workshop:
  - draft of roadmap for a data driven approach also in HLbL

- **goal**: realistic error estimate on $a_\mu$ / reduce to $2 \times 10^{-10}$ (20% of HLbL)